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AN INNOVATIVE APPROACH TO EVALUATE THE BEHAVIOUR OF VERTICALLY LOADED PILE GROUPS BASED ON ELASTIC THEORY

M. Cao[1], L.Z. Chen[2] and S.L. Chen[3]

ABSTRACT: An efficient analytical approach is proposed to calculate the settlement of a pile group under vertical loads. The proposed approach is based on the superposition of the displacement of individual pile. In the superposition calculation, an interaction factor, which was determined using the technique by Muki and Sternberg, is employed to facilitate the analysis of pile groups subjected to static vertical loads. The proposed interaction factor can consider the strengthening effect of intervening piles. The solution of the proposed approach is compared with other existing solutions. Their difference in estimating the behaviour of pile groups is investigated. Finally, numerical examples on two pile groups are presented to discuss the influence of dimensionless pile and soil parameters on the behaviour of pile groups. The results show that the proposed approach gives a more reasonable analysis on pile groups.

Keywords: Pile group, vertical load, interaction factor, settlement

INTRODUCTION

Over the years, the reliable prediction of pile group displacement at working load remains a major problem in civil engineering especially in lowland areas where pile groups are widely used. Static response of pile groups has been investigated by using a variety of empirical, analytical, and numerical techniques. Analysis of pile groups can be conducted in two ways: one is computer-based direct analysis of the whole group; another one is approximate solution using superposition of interaction factors.

Currently, the approach available for the direct analysis of axially loaded pile groups falls into four main categories: (1) simplified analytical methods involving separation of loads carried by shaft and base (e.g. Randolph, 1977); (2) integral equation methods (also known as the boundary element method), employing either load-transfer functions to represent the pile-soil interaction (e.g. Coyle & Reese, 1966; Kraft et al., 1981) or elastic continuum theory to represent the soil mass response (e.g. Butterfield & Banerjee, 1971; Banerjee, 1970; Poulos & Davis, 1980); (3) finite element methods (e.g. Desai, 1974; Valliappan et al., 1974; Balaam et al., 1975; Ottaviani, 1975; Jardine et al., 1986; Liang, 2003), in which a variety of constitutive soil models can be utilized, and such factors as soil nonhomogeneity and anisotropy can be taken into account; (4) the variational methods (Shen et al, 1997, 1999, 2000; Shen & Teh, 2002).

The direct analysis is preferable because it is accurate within the validity of the assumptions and provides extensive information. However, all the aforementioned methods have limitations. For example, a 3D finite element analysis is generally too expensive to get its application in the practical engineering. A criticism on the load transfer method is that the model does not take proper account of the continuity of the soil mass. The integral equation method is relatively accurate and rigorous; however, this method still requires a large amount of computer storage and subsequently long computational time for pile groups with large sizes.

In order to overcome these limitations, an efficient and straightforward approach for pile group analysis based on the principle of superposition and the interaction factor concept has been proposed (Poulos, 1968; Banerjee & Driscoll, 1978; Randolph & Wroth, 1979; Poulos & Davis, 1980; Caputo & Viggiani, 1984; Bilotta et al., 1991; Mandolini & Viggani, 1997; Guo & Randolph, 1999). This simplified method provides an efficient means of computing the pile-pile interaction. However, it should be mentioned that the prevailing interaction factor approaches, due to the neglect of the strengthening effect of intervening piles, enlarge the interaction factor and hence lead to an overestimation of

[1] Ph.D. Candidate, Shanghai Jiao Tong University, Department of Civil Engineering, Shanghai, CHINA
[2] Professor, Shanghai Jiao Tong University, Department of Civil Engineering, Shanghai, CHINA
[3] Associate Professor, Shanghai Jiao Tong University, Department of Civil Engineering, Shanghai, CHINA
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the settlement of pile group (El Sharnouby & Novak, 1990).

In this paper, the interaction factor for pile group analysis is reexamined and the aforementioned limitation for conventional calculation of the interaction factor is removed by using a fictitious pile method which was proposed by Muki and Sternberg (1970). Comparisons between the proposed approach and the available results are presented and their difference in predicting the behaviour of pile groups is investigated. Parametrical studies are also presented to describe the influence of the governing parameters on the load distribution and displacement of the pile group.

TWO-PILE INTERACTION ANALYSIS

In this section, a mathematical formulation is presented for the analysis of interaction factor between any two piles in a group under vertical loads P_0. As shown in Fig. 1, let $\{0, x, y, z\}$ be a rectangular Cartesian coordinate frame spanning the homogeneous semi-infinite elastic soil medium B. Two embedded piles in the group denoted by B_1' and B_2' are assumed to have the same length *L*, diameter *d*, and circular cross-sectional region Π_z ($0 < z < L$). The center-to-center spacing of the two piles is denoted by *S*.

Following the technique of Muki and Sternberg of these problems, the embedding soil medium is extended throughout the half space and two fictitious piles, B_{*1} and B_{*2}, are introduced at their original locations (see Fig. 1). The Young's modulus E_* of each fictitious pile is equal to the difference between those of the real pile and the extended soil, i.e.,

$$E_* = E_p - E_s \tag{1}$$

where E_p is the Young's modulus of the real piles, and E_s is the Young's modulus of soil.

We treat the extended soil B as a three-dimensional elastic continuum which is represented by material constants E_s and μ_s. In contrast, the two fictitious piles B_{*1} and B_{*2} are regarded as one-dimensional elastic continua as far as their constitutive laws and equilibrium conditions are concerned. Considering the identity of any two piles in the group, only the embedded pile 1 and the corresponding fictitious pile B_{*1} will be considered in the subsequent analysis. Apparently, B_{*1} is governed by the stress-strain relation

$$\frac{P_*(z)}{A} = E_* \varepsilon_*(z) \quad (0 \le z \le L) \tag{2}$$

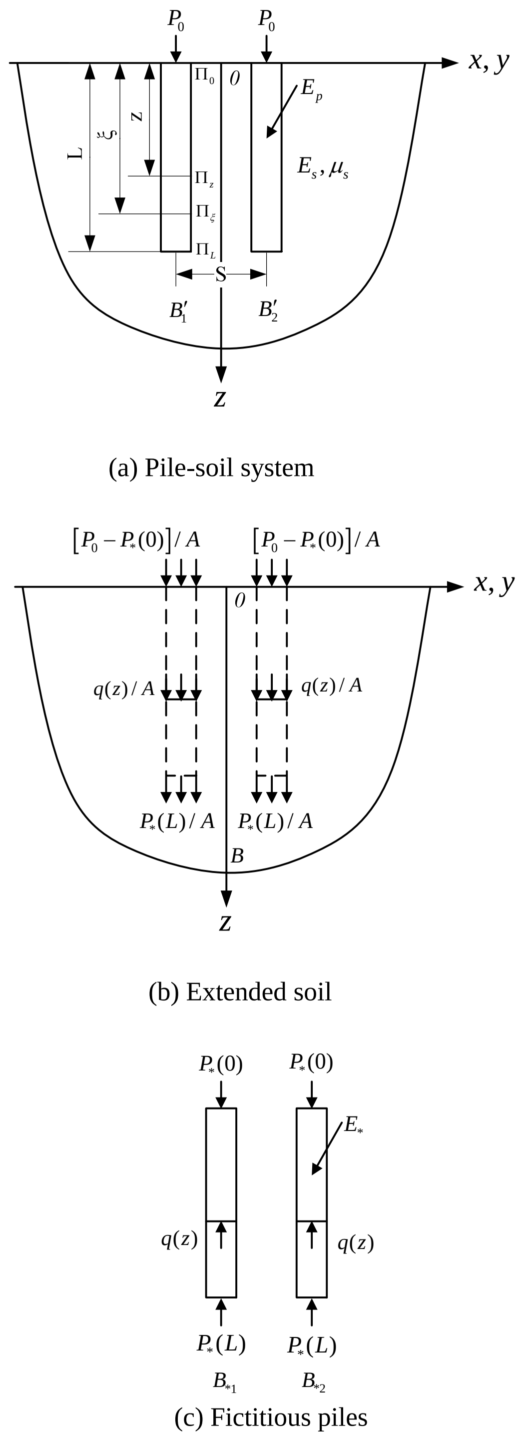


Fig. 1 The model of pile group

where A is the cross-sectional area of the pile, $P_*(z)$ is the scalar axial force of the fictitious pile B_{*1} and $\varepsilon_*(z)$ is the associated axial strain. Consideration of vertical equilibrium for B_{*1} yields the differential equation:

$$\frac{dP_*(z)}{dz} + q(z) = 0 \quad (0 \le z \le L) \tag{3}$$

where $q(z)$ is the "bond force" per unit pile-length exerted by the extended soil on the fictitious pile B_{*1} at depth z. The fictitious pile 1 is also subjected to the external axial force $P_*(0)$ and $P_*(L)$, concentrated within the terminal cross-sections Π_0 and Π_L, respectively. Here $P_*(0)$ is the portion of applied force $P(0)$ transmitted to the fictitious pile 1 directly, and $P_*(L)$ is the bond-force exerted by extended soil B on the fictitious pile 1. We adopt the requirement that the axial strain $\varepsilon_*(z)$ in the fictitious pile 1 be equal to that of the center point in the cross-section Π_z in the extended soil. Then the axial strain $\varepsilon_*(z)$ in the fictitious pile 1 can be written as

$$\begin{aligned}\varepsilon_*(z) = \frac{1}{A}\int_{\Pi_z}\sum_{j=1}^{2}\Big\{&\left[P_0 - P_*(0)\right]\hat{\varepsilon}_z^{(1,j)}(z,0)\\ &+P_*(L)\hat{\varepsilon}_z^{(1,j)}(z,L)\\ &+\int_0^L q_*(\xi)\hat{\varepsilon}_z^{(1,j)}(z,\xi)d\xi\Big\}dA\\ &(0 \le z \le L, 0 \le \xi \le L, z \ne \xi)\end{aligned} \tag{4}$$

where the strain influence functions $\hat{\varepsilon}_z^{(1,j)}(z,\xi)\left(j=1,2\right)$ represent respectively the vertical strains of the semi-infinite soil at depth z along the axis of real pile 1 due to a uniform circular load over the cross-section Π_ξ in the position of pile j, acting in the positive z-direction, and the resultant applied force having unit magnitude. The influence functions $\varepsilon_z^{(1,j)}(z,\xi)$ may be obtained by integrating Mindlin's solutions (Mindlin, 1936).

With the aid of Eqs. (2) and (3), Eq. (4) can be rewritten as

$$\begin{aligned}&\left[\frac{1}{E_*}+\frac{(1-2\mu_s)(1+\mu_s)}{E_s(1-\mu_s)}\right]P_*(z)\\ &-A\int_0^L P_*(\xi)\sum_{j=1}^{2}\frac{\partial\hat{\varepsilon}_z^{(1,j)}(z,\xi)}{\partial\xi}d\xi\\ &= AP_0\sum_{j=1}^{2}\hat{\varepsilon}_z^{(1,j)}(z,0)\\ &\quad(0 \le z \le L, 0 \le \xi \le L, z \ne \xi)\end{aligned} \tag{5}$$

Eq. (5) is a Fredholm integral equation of the second kind, and the solution of which furnishes the unknown function $P_*(z)$ (the axial force along the fictitious pile 1). This integral equation is readily amenable to a numerical solution. Once $P_*(z)$ has been found, the solution for the vertical displacement, $w(z)$, of pile 1 in the half space of the extended soil can be obtained as

$$\begin{aligned}w(z) &= P_0(0)\sum_{j=1}^{2}\hat{w}^{(1,j)}(z,0)\\ &\quad+\int_0^L P_*(\xi)\sum_{j=1}^{2}\frac{\partial\hat{w}^{(1,j)}(z,\xi)}{\partial\xi}d\xi\\ &\quad(0 \le z \le L, 0 \le \xi \le L)\end{aligned} \tag{6}$$

where $\hat{w}^{(1,j)}(z,\xi)\left(j=1,2\right)$ are the settlement influence functions having the similar definition with $\varepsilon_z^{(1,j)}(z,\xi)$ which can be obtained by integrating Mindlin's solutions.

On the other hand, for the analysis of the single pile case, we can rewrite Eqs. (5) and (6) as follows

$$\begin{aligned}&\left[\frac{1}{E_*}+\frac{(1-2\mu_s)(1+\mu_s)}{E_s(1-\mu_s)}\right]P_*'(z)\\ &-A\int_0^L P_*'(\xi)\frac{\partial\hat{\varepsilon}_z^{(1,1)}(z,\xi)}{\partial\xi}d\xi\\ &= AP_0\hat{\varepsilon}_z^{(1,1)}(z,0)\\ &\quad(0 \le z \le L, 0 \le \xi \le L, z \ne \xi)\end{aligned} \tag{7}$$

$$\begin{aligned}w'(z) &= P_0(0)\hat{w}^{(1,1)}(z,0)\\ &\quad+\int_0^L P_*'(\xi)\frac{\partial\hat{w}^{(1,1)}(z,\xi)}{\partial\xi}d\xi\\ &\quad(0 \le z \le L, 0 \le \xi \le L)\end{aligned} \tag{8}$$

where $P_*'(z)$ is the axial force along the fictitious pile of the single pile, and $w'(z)$ is the vertical displacement of the single pile.

At this stage the pile head displacements for both a single pile and a two-pile group have been formulated, with due account of the pile strengthening effect. Therefore, according to the definition by Poulos (1968), the interaction factor α between two identical piles under vertical loads is expressed as

$$\alpha = \frac{w(0)-w'(0)}{w'(0)} \tag{9}$$

where $w(0)$ and $w'(0)$ are the pile head displacements for a two pile group and a single pile, respectively. Several interaction factors are presented by Poulos & Davis (1980). However, the interaction factors are sometime overestimated. There are two reasons for this. Firstly, they were calculated ignoring the strengthening effect of intervening piles. Secondly, the number of elements was too small which may induce an error. For these reasons, the present method is employed to introduce an alternative interaction factor.

ANALYSIS OF PILE GROUPS

When there are similarly loaded neighboring piles, the overall displacement of a pile may be obtained by superimposing the individual displacements (Cooke, 1974), that is, for larger groups, the individual interaction factor may be superposed to yield the total settlement. Thus, for a group of n identical piles, the settlement $w_k(0)$ of any pile k in the group is given by superposition, as

$$w_k(0) = w'(0)\sum_{\substack{j=1\\ j\neq k}}^{n} P_{0j}\alpha_{kj} + w'(0)P_{0k} \qquad (k=1,2,...n) \tag{10}$$

where P_{0j} is the load on the head of pile j and α_{kj} is the interaction factor between piles k and j.

For a pile group with a rigid cap, all pile settlements are equal, i.e.,

$$w_k(0) = w_c \quad (k=1,2,...n) \tag{11}$$

where w_c is the settlement of pile group cap. Also, for vertical load equilibrium,

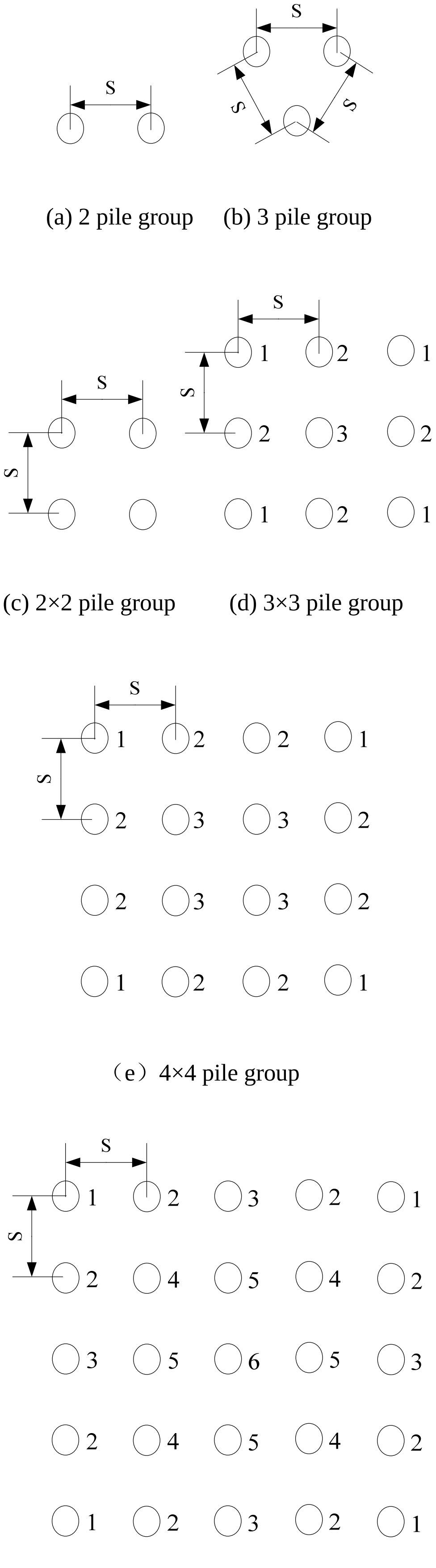
$$P_0 = \sum_{j=1}^{n} P_{0j} \tag{12}$$

Eqs. (11) and (12) give $n+1$ simultaneous equations that can be solved for the unknown load distribution on individual piles and the settlement of pile group.

COMPARISON WITH EXISTING SOLUTIONS

To investigate the strengthening effect of intervening piles, comparisons were made with the results obtained by Poulos (1968) and Butterfield & Banerjee (1971) for pile groups embedded in a homogeneous, isotropic linear elastic half-space. The pile cap is assumed to be rigid and not in contact with the ground.

The dimensionless parameters of interests here are $P_{0j}/(Gw_c d)$, P_{0j}/P_{av}, E_p/E_s, L/d and S/d, in which $G=$ the shear modulus of soil; P_{0j}/P_{av} is the normalized vertical load on pile head where the average load is calculated by $P_{av} = \sum_{j=1}^{n} P_{0j}/n$; $E_p/E_s=$ relative stiffness of pile to soil; L/d = slenderness ratio and S/d = pile spacing ratio. In the comparative numerical study, the following standard values are adopted: $E_p/E_s=\infty$, μ_s =0.5, S/d =2.5 and L/d =25. It



(a) 2 pile group (b) 3 pile group

(c) 2×2 pile group (d) 3×3 pile group

（e）4×4 pile group

（f）5×5 pile group

Fig. 2 Configurations for different pile groups

Table 1 Comparison of group reduction factor for friction pile groups with rigid cap

Type of group		2×2	3×3	4×4	5×5
R_G	Poulos (1968)	0.672	0.541	0.460	0.403
	Butterfield & Banerjee (1971)	0.665	0.550	0.456	0.396
	Present	0.645	0.496	0.419	0.365

Table 2 Comparison of load distribution P_{0j} / P_{av} in friction pile group with rigid cap

Type of group	Pile number	Poulos (1968)	Butterfield & Banerjee (1971)	Present
3×3	1	1.520	1.510	1.383
	2	0.740	0.750	0.811
	3	-0.050	-0.060	0.222
4×4	1	2.020	2.020	1.839
	2	0.960	0.965	0.987
	3	0.050	0.044	0.186
5×5	1	2.580	2.520	2.289
	2	1.180	1.190	1.260
	3	1.160	1.160	1.070
	4	0.010	0.048	0.196
	5	0.100	0.106	0.149
	6	0.190	0.095	0.099

should be noted that these values are used unless otherwise specified.

A comparison of the group reduction factor, R_G (the ratio of the settlement of the group to the settlement of a single pile carrying the same total load as the group) for 2×2, 3×3, 4×4 and 5×5 pile groups is shown in Table 1. Apparently Poulos's and Butterfield & Banerjee's group reduction factors R_G are greater than those from present formulation for floating piles. Table 2 shows the load distributions P_{0j} / P_{av} in pile groups of different configurations. The pile number is displayed in Fig. 2.

The results indicate that, for all the pile groups of different configurations with rigid cap, the loads on individual piles in pile groups from present method generally become more uniform than those from the other two methods.

The differences appear to be due to the pile interaction effects may be overestimated by Poulos, Butterfield and Banerjee. The limitation of their methods lie in that they did not properly account for the strengthening effect of the piles to the surrounding soil. The interaction factors being superimposed in the present method are calculated for any two piles in the group, considering the presence of the others and thus considering the strengthening

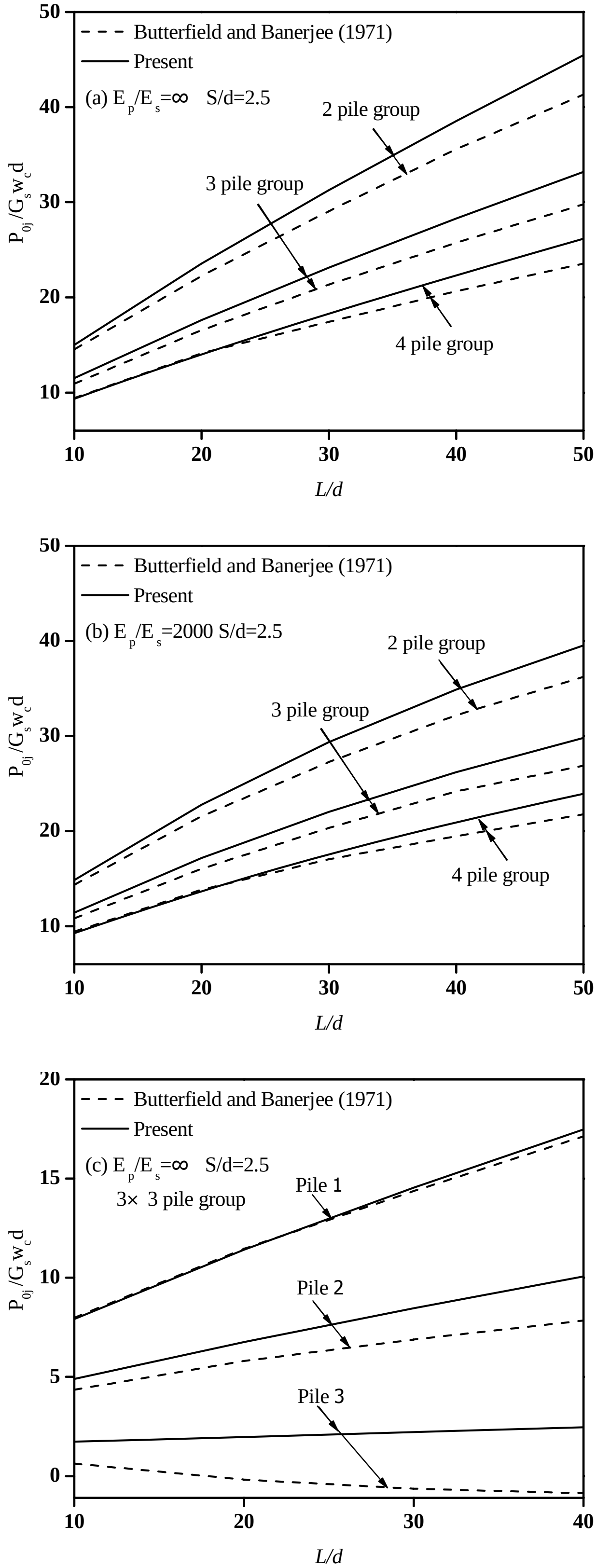


Fig. 3a Comparison of load distributions among individual piles in pile groups

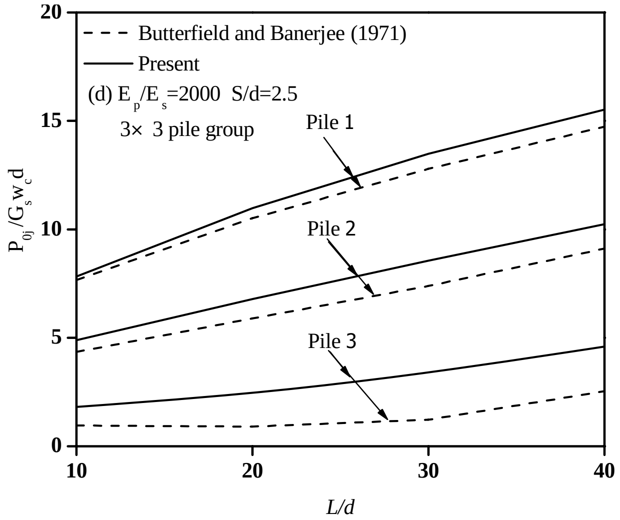


Fig. 3b Comparison of load distributions among individual piles in pile groups

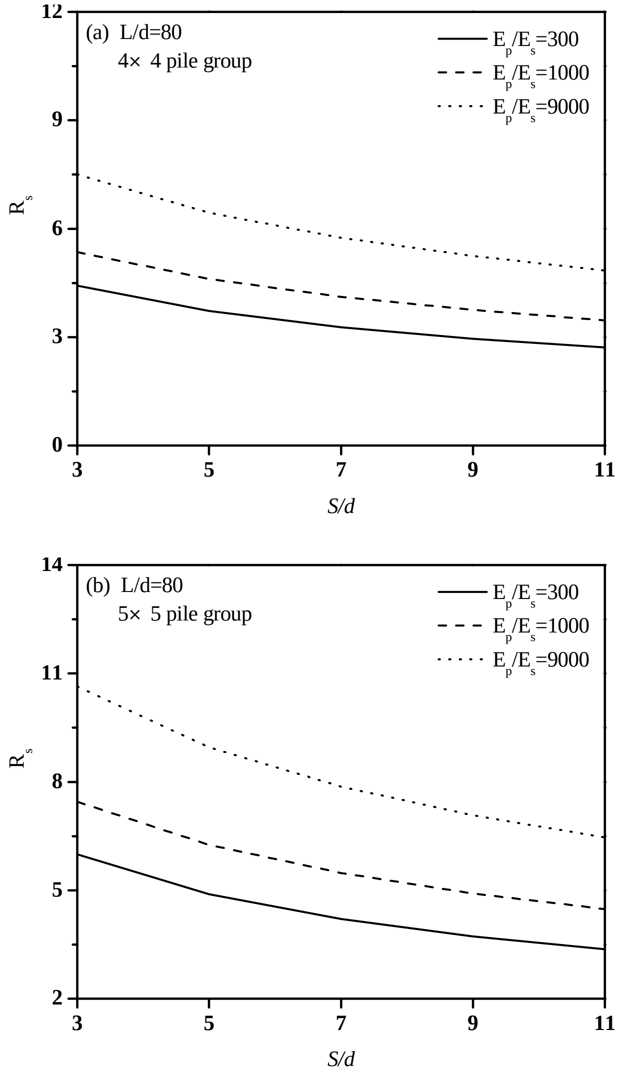


Fig. 4 Pile group settlement ratio versus pile spacing ratio

effect they have. The present approach, which predicts smaller settlements of pile group and more uniform load distributions compared to the existing solutions, may provide an alternative way to estimate the pile group behaviour for accuracy purpose.

Figures 3a and 3b show a comparison of the dimensionless stiffness $P_{0j}/\left(Gw_c d\right)$ of individual piles within floating pile groups of different configurations calculated for different slenderness ratios by Butterfield & Banerjee (1971) and the authors. The pile numbers are again shown in Fig. 2. It should also be noted that the present method consistently estimates a smaller settlement of pile group and a more uniform load distribution within pile groups of different configurations.

DISCUSSIONS

The behaviour of pile groups under vertical load are parametrically studied in this section. Particular attention is paid to settlement ratio R_s and load distribution P_{0j}/P_{av}. Here R_s is defined as the ratio of the vertical settlement of the pile group to that of a single pile subjected to the average individual pile load in the group. The reference values adopted here are L/d =80, S/d =4, μ_s =0.3, E_p/E_s =300, 1000 and 9000. In the following analysis, these values are used. The pile numbers for 4×4 and 5×5 pile groups are shown in Fig. 2.

Figure 4 displays settlement ratios R_s versus the pile spacing for three values of stiffness ratio, E_p/E_s, and for two pile groups of different sizes. As expected, the pile settlement ratio decreases monotonously with increasing pile spacing. The general trend of R_s varying with *S/d* is nearly independent of the E_p/E_s value. However, a decrease in stiffness ratio will greatly reduce the settlement ratio of the pile group. For example, as shown in Fig. 4 (a), the settlement ratio of the pile group with a stiffness ratio of 9000 may be reduced to approximately 69-78% of that of the pile group with E_p/E_s =300.

The settlement ratios against the pile slenderness ratio are plotted in Fig. 5. Again three values of stiffness ratio and two pile groups of different sizes are considered in the analysis. It can be observed that for both pile groups there is a poor agreement of the settlement ratios between the different stiffness ratios. With an increase of pile length, the settlement ratio tends to increase for a stiffness ratio of 9000, while the settlement ratio tends to decrease for a stiffness ratio of 300.

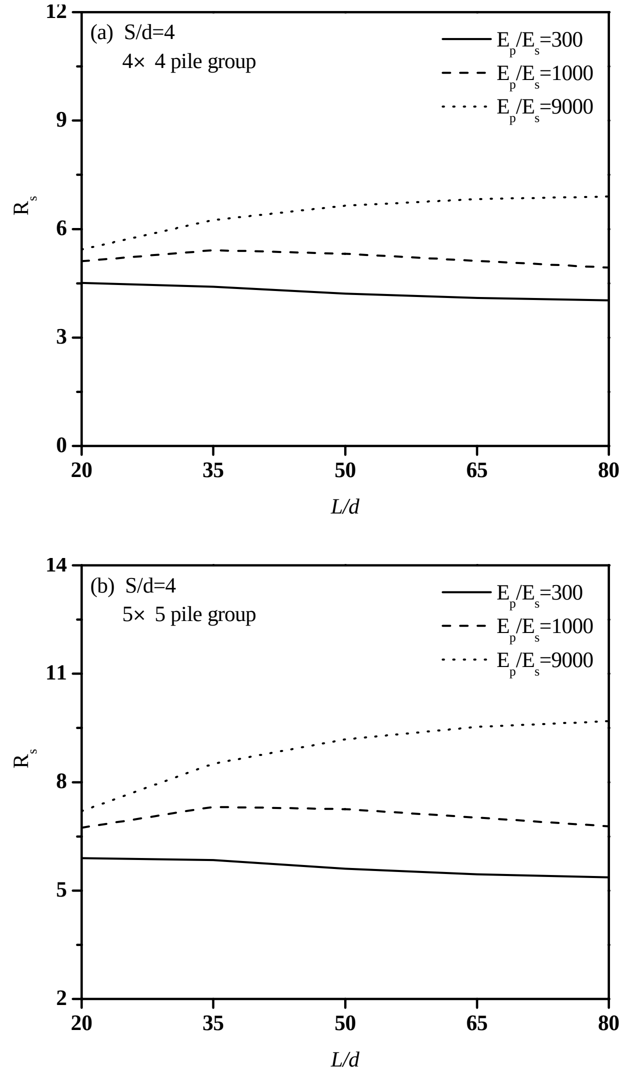


Fig. 5 Pile group settlement ratio versus pile slenderness ratio

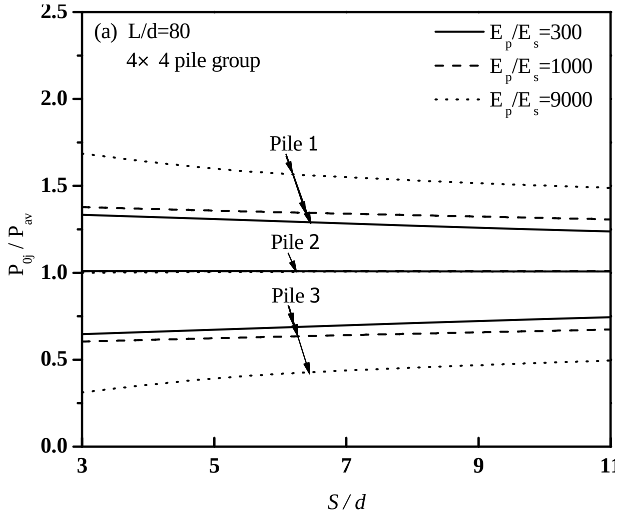


Fig. 6a Load distribution versus pile spacing ratio

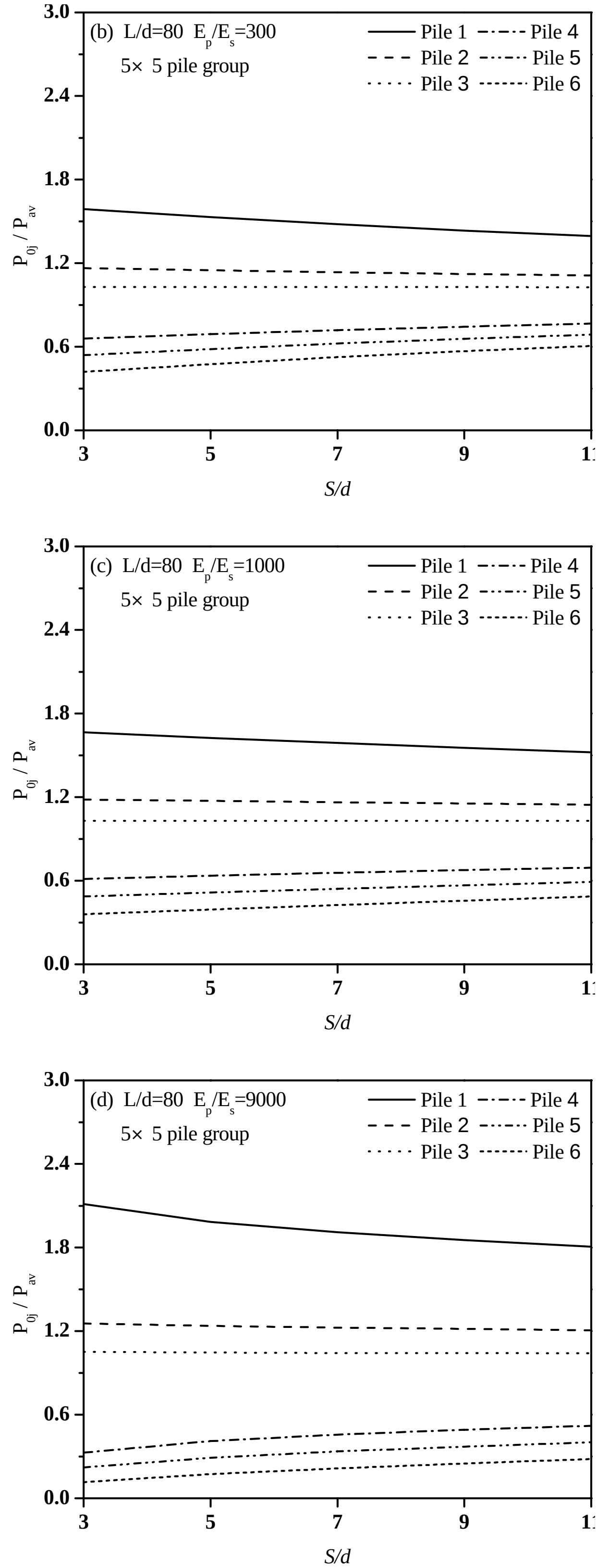


Fig. 6b Load distribution versus pile spacing ratio

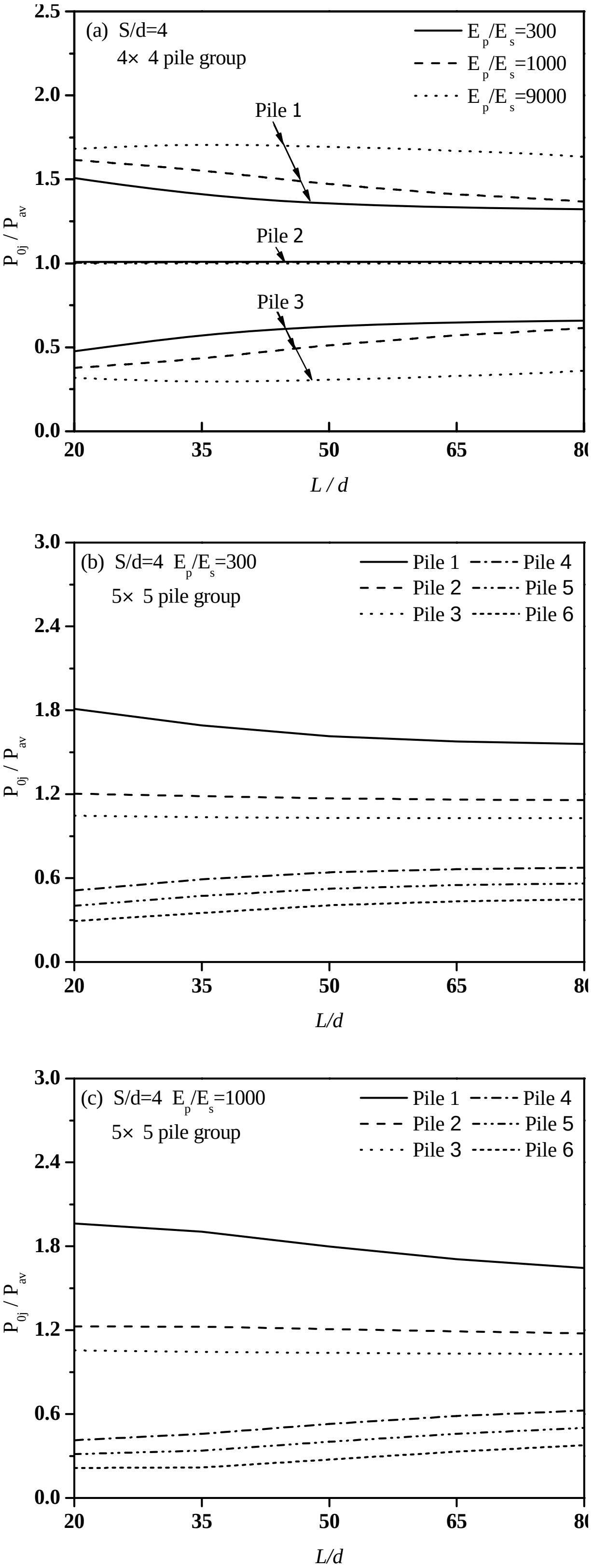


Fig. 7a Load distribution versus pile slenderness ratio

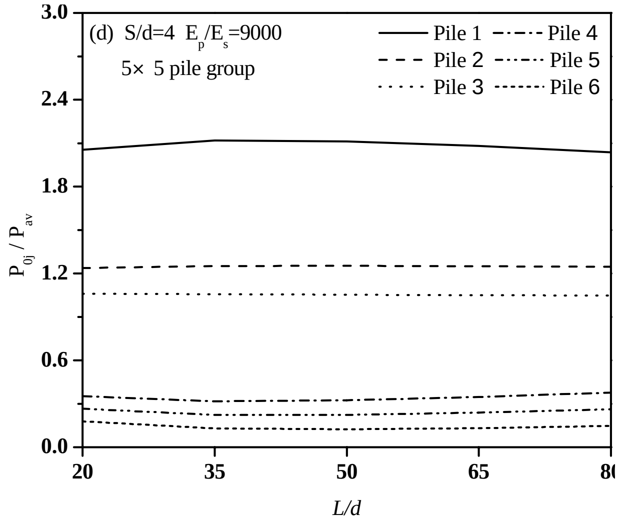


Fig. 7b Load distribution versus pile slenderness ratio

Figures 6a and 6b show the load distributions among individual piles versus the spacing ratio for three values of stiffness ratio in two pile groups of different size. The individual pile load is affected by the pile spacing. A small load is carried by the internal piles of closely-spaced group. As the pile spacing increases, the pile head loads tend to be uniform and the general shape of the curves is independent of the E_p / E_s values. However when the stiffness ratio becomes substantially rigid (E_p / E_s =9000), the pile head loads tend to be non-uniform.

The load distributions among the piles against pile length are plotted in Figs. 7a and 7b for three stiffness ratios and two pile groups. Figs. 7a and 7b indicate that for both pile groups there is a poor agreement of the curve shapes between the different stiffness ratios. With the increase of pile length, the load distributions on individual piles tend to be non-uniform for a stiffness ratio of 9000, however, the load distributions on individual piles tend to be uniform for a stiffness ratio of 300.

To investigate the efficiency of the proposed approach based on the superposition of individual pile displacement, comparison was made with the results obtained by a full analysis which is also based on the technique proposed by Muki and Sternberg (1970). The results of this study are shown in Fig. 8 for a 3×3 pile group of S / d =4, μ_s =0.3, E_p / E_s =1000. Fig. 8 (a) indicates that the dimensionless stiffness $P_{0j} / \left(G w_c d\right)$ by the present solution agrees closely with that by the full analysis. In Fig. 8 (b) the time ratio, m , which is defined as the ratio of the computational time of the present solution to that of the full analysis is plotted against the pile length. Fig. 8 (b) shows that the computational time of the present solution is about 20%

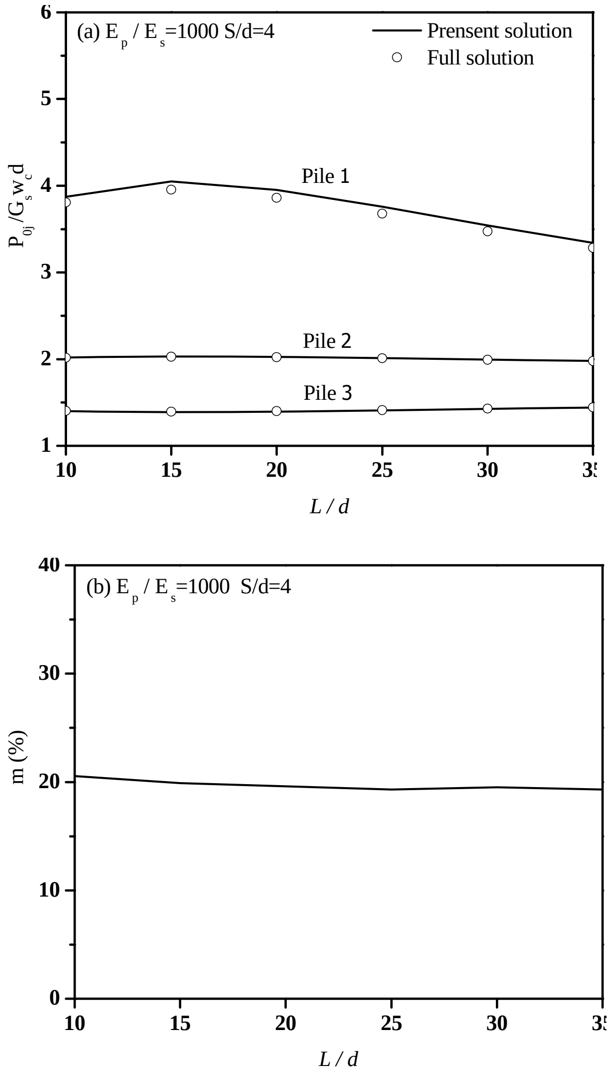


Fig. 8 Comparison of present solution with full solution

of that of the full analysis for various pile lengths, which indicates that the present approach is efficient compared to the full analysis method.

CONCLUSIONS

This paper is aimed to establish a simple, efficient approach to predict the settlement of pile groups. The interaction factor is used to predict the behaviour of pile groups embedded in a homogeneous isotropic elastic half-space. The current solutions have been compared with previous numerical analyse. Two pile groups with 4×4 and 5×5 piles have been analyzed. The main conclusions from this paper are drawn as follows:

(a) Compared to the full rigorous analytical method, the proposed method save much computational time. For the analyzed numerical example, the computational time of the proposed approach is only about 20% of that of a full analysis. Therefore the proposed method is efficient and more easily applicable to engineering practice.

(b) The interaction factor in the proposed method, which takes the strengthening effect of intervening piles into account, gives a more reasonable solution to the settlement of pile group and load distribution on individual piles. In the proposed solution, the calculated settlement of pile group is smaller and the load distribution on individual piles is more uniform compared to those of the existing solutions.

(c) The results of the parametric study reveal that with the increase of the pile length, the settlement ratio trends to increase for a stiffness ratio of 9000. However, for a stiffness ratio of 300 with the increase of the pile length the settlement ratio trends to decrease. The results of parametric study also reveal that pile-soil stiffness ratio plays an important role on the load distributions among individual piles for both pile groups. With an increase of the pile length, under the condition of E_p / E_s =9000 the load distribution on individual piles becomes less even, whereas, under the condition of E_p / E_s =300 the load distribution on individual piles becomes more even.

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