The Effect of Using Different Elastic Moduli on Vibration of Laminated CFRP Rectangular Plates

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Abstract

This paper deals with effects of using different sets of material constants on the natural frequencies of laminated composite rectangular plates. The plate is symmetrically laminated by thin layers composed of recently developed carbon fiber reinforced plastic (CFRP) materials. Numerical experiments are conducted by using a semi-analytical solution based on the thin plate theory and the lamination theory. The displacements are assumed to accommodate any combination of classical boundary conditions. The material property is expressed by a set of four elastic constants, and some typical sets of values are cited from the recent literature. Furthermore, a new standard set of discretized constants is proposed to uncover the underlying characteristics of the existing constants. The convergence study is carried out first, and the lowest five natural frequencies are calculated for five sets of classical boundary conditions including totally free through totally clamped cases. Next, a new definition of frequency parameters is introduced to promote more physically meaningful comparison among the obtained results, and the effect of using slightly different constants is clarified for unified comparison and insights. It is also discussed to derive approximate frequency formulas by linear regression analysis and to test accuracy of the formulas.

Keywords: Carbon fiber reinforced plastic; laminated composite plate; material constants; natural frequency; vibration

1. Introduction

Structural components in the form of flat plates are found practically in all fields of industry, and the literatures related to free vibration of flat plates are numerous. Leissa [1] compiled in 1969 a monograph "Vibration of Plates" and it is still cited in many papers. For isotropic rectangular plates, he published a comprehensive set of natural frequencies for all possible combinations of classical boundary conditions [2]. Since in the 1970's, advanced fibrous composite materials have been developed and served as plate components in many industrial applications, such as airplane, automobile and marine structures. Generally, fiber reinforced plastics have microstructures made of reinforcing fiber and matrix material. There are various fibers, ranging from natural fibers to chemical fibers such as glass, boron and carbon fibers. When such constituents merge into one material, it shows anisotropic characteristics. Among various types of fibers, the use of carbon fibers in reinforced plastics (CFRP) is becoming more increasingly dominant in weight-sensitive structures. For the reason, some books [3-5] dealing with mechanics of the composites have been published to serve for analysis and design, and review papers have appeared, for example [6].

Since the publication of journal papers in the 1970's, researchers such as Bert [7] published papers on vibration of laminated composite plates, and it has led to improvement of lamination theories to the first order and higher order plate theories [5, 8]. For rectangular plates with arbitrary edges, combination of classical boundary conditions became possible to be analyzed [9] in good accuracy. Application of the finite element method has also been active in analyzing laminated plates [10, 11].

For defining elastic problem of laminated CFRP plates, four independent elastic constants are necessary, unlike isotropic plates with only two independent constants being needed. Previous literatures have used different values of the constants, because the improvement of fiber stiffness is in progress, and many past researchers have used elastic constants obtained from measurement tests supplied from different chemical companies. This fact makes direct comparison difficult among the published results, and was also obstacles preventing from compiling design date book.

Despite such practical needs, however, there is no literature to discuss the influence caused by using different elastic constants. The present study takes up this problem for studying the effect when slightly different constants are used in the vibration analysis of laminated CFRP plates. In doing so, numerical experiments are conducted

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to calculate natural frequencies of laminated CFRP rectangular plates by using various sets of constants [3, 12-14], and for effective insights, new standard constants are proposed to understand vibration behaviors in the comprehensive way.

2. Analytical Method

Figure 1 shows a laminated rectangular plate in the coordinate system and in each layer the major and minor principal axes are denoted by the L and T axes. The dimension of the whole plate is given by $a \times b \times h$ (thickness). The plate considered is limited to symmetric laminate, and the total number of layers is defined as $2N$ (i.e., N layers in the upper(lower) half cross-section).

Free vibration of a macroscopic model for such thin symmetric plates is governed in the classical lamination (plate) theory by

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4}$$
$$+4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1)$$

where w is a deflection and ρ is a mean mass per unit volume of the plate. The D_{ij} (i, j = 1, 2, 6) are the bending stiffness of the symmetric laminate defined by

$$D_{ij} = \frac{2}{3}\sum_{k=1}^{N}\bar{Q}_{ij}^{(k)}\left(z_k^3 - z_{k-1}^3\right) \qquad (2)$$

with z_k being a thickness coordinate measured from the mid-surface and $\bar{Q}_{ij}^{(k)}$ being elastic constants [3, 4] in the k-th layer, obtained from

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\nu_{TL}}, Q_{12} = \frac{E_L \nu_{TL}}{1-\nu_{LT}\nu_{TL}},$$
$$Q_{22} = \frac{E_T}{1-\nu_{LT}\nu_{TL}}, Q_{16} = \frac{G_{LT}}{1-\nu_{LT}\nu_{TL}}$$
$$(3)$$

(superscript (k) is omitted) by considering a fiber orientation angle θ_k in the layer. The E_L and E_T are moduli of longitudinal elasticity in the L and T directions, respectively, G_{LT} is a shear modulus and ν_{LT} is a Poisson ratio.

For the small amplitude (linear) free vibration, the deflection w of a thin plate may be written by

$$w\left(x,y,t\right) = W\left(x,y\right)\sin \omega t \qquad (4)$$

where W is the amplitude and ω is a radian frequency. Natural frequency is normalized as a frequency parameter

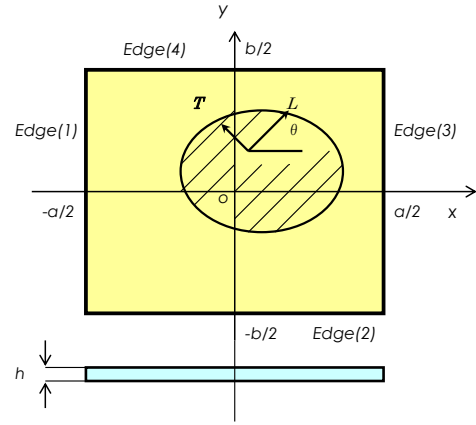$$\Omega = \omega a^2 \left(\rho h / D_0\right)^{1/2} \qquad (5)$$

Figure 1. Laminated rectangular plat

where ω is a radian frequency of the laminated plate and D_0 is a reference bending stiffness.

Then, the maximum strain energy due to the bending is expressed by

$$U_{max} = \frac{1}{2}\iint_A \{\kappa\}^T \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \{\kappa\} dA \qquad (6)$$

and $\{\kappa\}$ is a curvature vector

$$\{\kappa\} = \left\{ -\frac{\partial^2 W}{\partial x^2} \quad -\frac{\partial^2 W}{\partial y^2} \quad -2\frac{\partial^2 W}{\partial x \partial y} \right\}^T \qquad (7)$$

The maximum kinetic energy is given by

$$T_{max} = \frac{1}{2}\rho h \omega^2 \iint_A W^2 dA \qquad (8)$$

In the Ritz method the amplitude is assumed in the form

$$W\left(x,y\right) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} A_{mn} X_m\left(x\right) Y_n\left(y\right) \qquad (9)$$

where A_{mn} are unknown coefficients, and $X_m(x)$ and $Y_n(y)$ are the functions modified so that any kinematical boundary conditions are satisfied at the edges with applying "boundary indices" [9, 15, 16]. To simplify the analysis, the normalized co-ordinate system O-$\xi\eta$ ($-1 \leq \xi$, $\eta \leq 1$) is defined in equation

$$\xi = 2x/a, \ \eta = 2y/b \qquad (10)$$

and $X_m(\xi)$ and $Y_n(\eta)$ are the displacement functions

$$X_m\left(\xi\right) = \xi^m \left(\xi+1\right)^{bc1}\left(\xi-1\right)^{bc2}$$
$$Y_n\left(\eta\right) = \eta^n \left(\eta+1\right)^{bc3}\left(\eta-1\right)^{bc4} \qquad (11)$$

The bci ($i = 1, 2, 3$ and 4) is the boundary index which is used to satisfy the kinematical boundary conditions on each edge, and they are defined as

$$bc1, bc2, bc3, bc4 = \begin{cases} 0 & \text{(free)} \\ 1 & \text{(simple support)} \\ 2 & \text{(clamp)} \end{cases} \quad (12)$$

on each of four edges.

After substituting Eq.(9) into the sum of energies (6) and (8), the stationary value is obtained by

$$\frac{\partial}{\partial A_{\overline{mn}}}\left(T_{max} - U_{max}\right) = 0$$

$$\left(\overline{m} = 0,1,..,M-1; \overline{n} = 0,1,..,N-1\right) \quad (13)$$

The minimizing process gives a set of linear simultaneous equations in terms of the coefficients A_{mn}, and the eigenvalues Ω may be extracted by using existing computer subroutines. This analytical procedure is a standard routine of the Ritz method but the special form of polynomials (13) can satisfy any kinematical boundary conditions [9, 15, 16].

3. Numerical Examples and Accuracy of Solution

3.1. Numerical examples

Symmetrically laminated square (a/b=1) and rectangular plates (a/b=2) are considered as numerical examples. The number of layers is eight and a lamination sequence is denoted by $[\theta_1/\theta_2/\theta_3/\theta_4]$s, where θ_1 is a fiber orientation angle of the outer most layer measured from x axis (as shown in Fig.1) and θ_4 is the angle of inner most layer. In the examples, the lamination sequence is limited to two typical types:

Cross-ply plate [0°/90°/0°/90°]s, (i.e., [(0°/90°)$_2$]s),

Angle-ply [30°/-30°/30°/-30°]s, ([(30°/-30°)$_2$]s)

(s: symmetric lamination), (hereafter, "°" is omitted). The boundary conditions are denoted by four capital letters, such as CSFF, labelling counter-clockwise starting from Edge(1) in Fig. 1. Here, "C" denotes a clamped edge (i.e., deflection and rotation are both rigidly constrained), "S" does a simply supported edge (deflection is constrained but bending moment is zero) and "F" does a free edge (bending moment and shear force are both zero). By using this notation, natural frequencies for SSSS (totally simply supported plate) are presented in both Table 3 and 4, and those for FFFF (totally free plate), CFFF (cantilever plate), CSFF (combination of C,S,F) and CCCC (totally clamped plate) are presented for square planform (a/b=1) in Tables 5, 6, 7 and 8, respectively. Results of rectangular plates (a/b=2) are in Tables 9 and 10.

3.2. Elastic constants of different CFRP materials

For a fixed geometry (a,b,h) and boundary condition (C,S,F) of plate, it is easily understandable that frequency parameters are governed only by the values of elastic constants, and it is the main topic of this work to study the effect from the constants. As shown in Eq. (3), there are four independent elastic constants in Q_{ij}: E_L and E_T being modulus of longitudinal elasticity (Young's modulus) in

the L (fiber) and T directions, respectively, G_{LT} being a shear modulus and a Poisson's ratio ν_{LT}, and there is a relation of $E_L/\nu_{LT} = E_T/\nu_{TL}$ among the constants.

The survey by the authors showed that a little different value of elastic constants has been employed previously, as listed in Table 1. It is observed that the difference is mainly caused by values of E_L but not by E_T and G_{LT}, because Young modulus of the fibers is a key factor in controlling the stiffness in the longitudinal direction within lamina, while the matrix material is almost the same, mostly epoxy material. In the table, Mat.1 indicates values used by Stanford and Jutte [12] in NASA and shows the lowest anisotropy rate E_L/E_T=11.6 among the listed sets of constants. Mat.2 and 3 are medium values listed in [3] and [13], respectively. Mat.4 indicates those used by Panesar and Weaver [14], a group in Bristle University, and it shows the highest anisotropy E_L/E_T=18.7. Finally, Mat.5 is introduced to represent the mean values of recently used CFRP materials and used as the standard reference for comparison. This set of discretized values is fictious and not measured in experiment, but it will be shown that it is useful as the representative constants for CFRP. When one measures deviation of listed materials from Mat.5, the differences in the constants show ranges of

Table 1. Elastic constants of carbon fiber reinforced plastic (CFRP) materials

	E_L [Gpa]	E_T [Gpa]	(E_L/E_T)	G_{LT} [Gpa]	ν_{LT}
Mat.1 [12]	128	11	(11.6)	4.5	0.25
Mat.2 [3]	138	8.96	(15.4)	7.1	0.30
Mat.3 [13]	139	8.76	(15.9)	4.57	0.32
Mat.4 [14]	168.980	9.050	(18.7)	5.00	0.288
Mat.5	150.0	10.0	(15.0)	5.00	0.30

Table 2. Convergence of the solution for cross-ply and angle-ply square plates (Mat.5)

Number of term					
m × n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Cross-ply plate [(0/90)$_2$]s, CSFF					
6x6	12.55	40.36	73.06	90.49	119.9
8x8	12.55	40.36	73.06	90.48	118.7
10x10	12.55	40.36	73.06	90.48	118.7
Cross-ply plate [(0/90)$_2$]s, SSSS					
6x6	42.55	102.1	134.5	170.2	217.2
8x8	42.55	102.1	134.5	170.2	213.9
10x10	42.55	102.1	134.5	170.2	213.8
Exact sol.	42.55	102.1	134.5	170.2	213.8
(m,n)	(1,1)	(1,2)	(2,1)	(2,2)	(1,3)
Angle-ply plate [(30/-30)$_2$]s, CCCC					
6x6	88.50	143.7	209.1	225.8	273.9
8x8	88.50	143.7	209.1	225.6	273.8
10x10	88.50	143.7	209.1	225.6	273.8

-15% (Mat.1)$\leqq$0% (Mat.5)$\leqq$15%(Mat.4) for E_L,
-12%(Mat.3)$\leqq$0%(Mat.5)$\leqq$11%(Mat.1) for E_T,
-10%(Mat.1)$\leqq$0%(Mat.5)$\leqq$42%(Mat.2) for G_{LT}, and
-17%(Mat.1)$\leqq$0%(Mat.5)$\leqq$7%(Mat.3) for ν_{LT}.

The following comparison is made by a frequency parameter, but two types of reference bending rigidity are used as

$$\Omega = \omega a^2 \left(\rho h / D_0 \right)^{1/2}, \; D_0 = E_T h^3 / 12\left(1 - \nu_{LT}\nu_{TL}\right) \text{ (14)}$$

with E_T, ν_{LT} for each material in Table 1, and

$$\Omega^* = \omega a^2 \left(\rho h / D_0^* \right)^{1/2}, \; D_0^* = E_T^* h^3 / 12\left(1 - \nu_{LT}^*\nu_{TL}^*\right) \text{ (15)}$$

with E_T^* =10 GPa, ν_{LT}^* =0.3. Note $\nu_{TL}^* = \left(E_T^* / E_L^*\right)\nu_{LT}^*$ of Mat.5, for global comparison among different materials listed in the table.

3.3. Convergence and comparison of the solution

Before the natural frequencies are compared to discuss the discrepancy stemming from different CFRP materials, validity of the solution should be established. Table 2 presents frequency parameters of laminated square plates with Mat.5 (i.e., $D_0 = D_0^*$ in Es.(14)(15)). Number of terms M, N in Eq.(9) is increased from six to ten for cross-ply (CSFF, SSSS) and angle-ply (CCCC) plates. They converge well within four significant figures.

For a special case of cross-ply (i.e., $D_{16} = D_{26} = 0$) simply supported rectangular plate (SSSS), the exact solution can be written in the form

$$\Omega = \pi^2 \left[\left(\frac{D_{11}}{D_0}\right)m^4 + 2\left(\frac{b}{a}\right)^2 \left(\frac{D_{12} + 2D_{66}}{D_0}\right)m^2 n^2 \right.$$
$$\left. + \left(\frac{b}{a}\right)^4 \left(\frac{D_{22}}{D_0}\right)n^4 \right]^{1/2} \quad \text{(16)}$$

with m and n being half wave numbers in x and y direction, respectively. The results from this formula exactly agree with the present solution within four significant figures in Table 2. All the frequency parameters listed hereafter will be obtained by using 10×10 solution.

4. Results and Discussions

4.1. Unified parameter for different CFRP

Table 3 presents lowest five frequency parameters Ω, defined in Eq.(14) by using E_T of each material, for simply supported (SSSS) square plate. The results are given for Mat.1 ~ Mat.5, and the differences （％） in frequency measured from Mat.5 are also written in the table. The frequency parameter Ω is non-dimensional and it does not include the plate geometry (a, b, h) and mass density ρ. So for the fixed geometry and density, values of the frequency parameters depend on ω itself and E_T and ν_{LT}.

From observation in the table, no significant differences （％） exist among the frequency orders (i.e.,

from the first to fifth frequency) and between the lamination schemes of cross-ply and angle-ply. The increasing (or decreasing) trend of the frequencies is unclear with respect to the constant E_L, which should be the most influential factor on the frequency values. Next, the degree of influence by E_L will be quantitatively focused.

Table 3. Frequency parameters Ω (Eq.(14)) of symmetric 8-layer square plates (SSSS)

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Cross-ply [(0/90)₂]s					
Mat.1	37.92	91.53	119.2	151.7	191.8
(%)	−10.9	−10.3	−11.4	−10.9	−10.3
Mat.2	44.31	105.3	137.8	177.2	218.5
(%)	4.1	3.2	2.5	4.1	2.2
Mat.3	43.67	104.6	138.2	174.7	219.1
(%)	2.6	2.5	2.7	2.6	2.5
Mat.4	46.76	111.9	149.0	187.0	234.9
(%)	9.9	9.7	10.8	9.9	9.9
Mat.5	42.55	102.1	134.5	170.2	213.8
Angle-ply [(30/−30)₂]s					
Mat.1	46.13	91.04	127.7	156.8	185.4
(%)	−11.0	−10.0	−11.4	−9.2	−10.9
Mat.2	52.71	103.7	147.0	178.4	212.2
(%)	1.8	2.5	2.0	3.3	2.0
Mat.3	53.16	103.60	148.1	176.6	213.5
(%)	2.6	2.4	2.7	2.3	2.6
Mat.4	57.28	110.7	159.9	187.6	229.9
(%)	10.6	9.5	10.9	8.6	10.5
Mat.5	51.80	101.1	144.1	172.7	208.1

Table 4. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer square plates (SSSS)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply [(0/90)₂]s					
Mat.1	39.76	95.97	125.0	159.0	201.1
(%)	−6.6	−6.0	−7.1	−6.6	−5.9
Mat.2	41.94	99.70	130.4	167.7	206.8
(%)	−1.5	−2.3	−3.0	−1.5	−3.3
Mat.3	40.89	97.96	129.4	163.5	205.1
(%)	−3.9	−4.0	−3.8	−3.9	−4.0
Mat.4	44.45	106.4	141.7	177.8	223.3
(%)	4.5	4.2	5.3	4.5	4.4
Mat.5	42.55	102.1	134.5	170.2	213.8
Angle-ply [(30/−30)₂]s					
Mat.1	48.37	95.45	133.8	164.4	194.4
(%)	−6.6	−5.6	−7.1	−4.8	−6.6
Mat.2	49.89	98.11	139.2	168.8	200.8
(%)	−3.7	−3.0	−3.5	−2.3	−3.5
Mat.3	49.77	96.96	138.6	165.4	200.0
(%)	−3.9	−4.1	−3.8	−4.3	−3.9
Mat.4	54.45	105.2	152.0	178.3	218.5
(%)	5.1	4.1	5.5	3.2	5.0
Mat.5	51.80	101.1	144.1	172.7	208.1

Table 5. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer square plates (FFFF)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]s$				
Mat.1	15.76	48.72	58.25	67.64	74.65
(%)	−5.2	−6.0	−5.7	−7.3	−6.9
Mat.2	19.66	49.62	63.55	69.93	80.21
(%)	18.3	−4.2	2.9	−4.1	0.0
Mat.3	15.89	49.71	59.23	70.17	77.06
(%)	−4.4	−4.1	−4.2	−3.8	−3.9
Mat.4	16.64	54.28	63.89	77.12	84.04
(%)	0.1	4.80	3.4	5.7	4.8
Mat.5	16.62	51.82	61.79	72.93	80.18
Angle-ply	$[(30/-30)_2]s$				
Mat.1	23.67	33.64	60.88	70.96	73.45
(%)	−1.4	−6.4	−6.9	−2.7	−5.9
Mat.2	25.90	35.14	64.11	74.19	77.31
(%)	7.9	−2.3	−1.9	1.8	−1.0
Mat.3	22.78	34.52	62.83	69.40	74.89
(%)	−5.1	−4.0	−3.9	−4.8	−4.1
Mat.4	23.64	37.79	68.58	73.23	81.65
(%)	−1.5	5.1	4.9	0.5	4.6
Mat.5	24.00	35.95	65.37	72.90	78.09

Table 6. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer square plates (CFFF)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]s$				
Mat.1	10.63	14.34	53.23	66.61	71.40
(%)	−7.3	−6.4	−6.0	−7.3	−7.0
Mat.2	10.99	16.22	55.72	68.86	75.94
(%)	−4.1	5.9	−1.6	−4.1	−1.1
Mat.3	11.02	14.70	54.29	69.10	73.80
(%)	−3.8	−4.0	−4.1	−3.8	−3.9
Mat.4	12.12	15.83	59.03	75.95	80.66
(%)	5.8	3.4	4.3	5.8	5.1
Mat.5	11.46	15.31	56.60	71.82	76.77
Angle-ply	$[(30/-30)_2]s$				
Mat.1	9.212	22.31	46.28	60.04	78.55
(%)	−7.3	−6.7	−5.5	−6.9	−6.2
Mat.2	9.740	23.25	48.83	62.89	82.95
(%)	−1.9	−2.7	−0.4	−2.4	−1.0
Mat.3	9.553	22.96	46.94	61.96	80.35
(%)	−3.8	−4.0	−4.2	−3.9	−4.1
Mat.4	10.46	25.16	50.92	67.71	87.45
(%)	5.3	5.3	3.9	5.0	4.4
Mat.5	9.933	23.90	49.00	64.46	83.76

Table 7. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer square plates (CSFF)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]s$				
Mat.1	11.68	37.95	67.81	84.51	111.6
(%)	−6.9	−6.0	−7.2	−6.6	−6.0
Mat.2	12.54	39.93	70.64	91.10	114.7
(%)	−0.1	−1.1	−3.3	0.7	−3.4
Mat.3	12.06	38.72	70.28	86.91	113.9
(%)	−3.9	−4.1	−3.8	−4.0	−4.1
Mat.4	13.16	42.08	77.12	94.42	124.1
(%)	4.9	4.3	5.6	4.4	4.6
Mat.5	12.55	40.36	73.06	90.48	118.7
Angle-ply	$[(30/-30)_2]s$				
Mat.1	14.42	36.15	66.51	80.68	110.1
(%)	−6.7	−5.5	−6.6	−4.6	−6.6
Mat.2	15.03	37.82	70.11	83.64	114.7
(%)	−2.8	−1.2	−1.6	−1.1	−2.7
Mat.3	14.85	36.67	68.40	80.90	113.3
(%)	−3.9	−4.2	−4.0	−4.3	−3.9
Mat.4	16.27	39.79	74.56	87.13	123.9
(%)	5.2	4.0	4.7	3.0	5.1
Mat.5	15.46	38.27	71.23	84.56	117.9

Table 8. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer square plates (CCCC)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]s$				
Mat.1	85.47	154.8	196.1	240.3	277.2
(%)	−6.7	−6.2	−7.1	−6.7	−6.0
Mat.2	88.65	159.7	203.7	250.9	284.5
(%)	−3.3	−3.2	−3.5	−2.6	−3.6
Mat.3	88.08	158.4	203.0	247.5	283.0
(%)	−3.9	−4.0	−3.8	−3.9	−4.0
Mat.4	96.31	172.6	222.6	270.0	308.4
(%)	5.1	4.6	5.5	4.9	4.6
Mat.5	91.64	165.0	211.1	257.5	294.9
Angle-ply	$[(30/-30)_2]s$				
Mat.1	82.66	135.8	194.2	214.9	255.8
(%)	−6.6	−5.5	−7.1	−4.7	−6.6
Mat.2	85.84	140.2	202.4	221.4	265.2
(%)	−3.0	−2.4	−3.2	−1.9	−3.2
Mat.3	85.04	137.7	201.1	215.9	263.1
(%)	−3.9	−4.1	−3.8	−4.3	−3.9
Mat.4	92.85	149.2	220.4	232.4	287.3
(%)	4.9	3.8	5.4	3.0	4.9
Mat.5	88.50	143.7	209.1	225.6	273.8

Therefore, use of Ω^* defined in Eq. (15) is considered next to make effective comparison and to provide with clear physical interpretation. Table 4 is in the same format as Table 3 for the same square plate, except that a new reference stiffness

$D_0^*=E_T^*h^3/12(1-\nu_{LT}^*\nu_{TL}^*)$ (E_T^*, ν_{LT}^* for Mat.5) is used for all the materials (i.e, even for Mat.1,2,3 and 4). Therefore, the frequency parameters for Mat.5 are identical as those in Table.3.

The idea here is that by using the identical elastic constants E_T^* and ν_{LT}^* in the frequency parameter, the differences are caused only by ω in the new frequency parameters. Actually, it is observed in Tables 4-8 that the increasing order of E_L (Mat.1<Mat.2 $\doteqdot$ Mat.3<Mat.4) among the materials basically reflects the order in the difference, except for FFFF plate, and the differences only for Mat.4 are positive, i.e. session Ω^* (Mat.5)< Ω^* (Mat.4) due to E_L (Mat.5)< E_L (Mat.4), unlike unnatural findings in Table 3.

The frequency parameters listed in Tables 5 (FFFF), 6 (CFFF), 7 (CSFF) and 8 (CCCC) are based on the same idea, and Ω^* uses the identical values of E_T^*=10GPa and ν_{LT}^*=0.3. In Table 5 (FFFF), it is seen that the differences for Mat.2 take alternatively positive and negative values, but those of Mat 3 show only all negative differences. This unpredicted behavior may be caused by the fact that in the FFFF case, there are three rigid body motions (elastic frequencies become zero) and the vibrations take complicated mode shapes to satisfy the self-equilibrium. Such strange behavior is found when the plate edges involve a number of free edges, such as FFFF, CFFF and CSFF, and it decreases as the number of free edges (F) diminished.

For Mat.1 with E_L being the smallest in Table 1, all the differences are negative in Tables 4-8 for ten cases (cross-ply and angle-ply plates with 5 different combinations of boundary conditions), and the average differences of the lowest five Ω for the ten cases stay within the range of -6.8~-4.6%. Similarly, for Mat.5 with E_L being the largest in the table, all the differences are positive for the ten cases.

By using the data in Tables 4 and 8, Figure 2 presents comparison of percentage differences for two lamination types (cross-ply, angle-ply) and two sets of boundary conditions (SSSS, CCCC). The lowest five frequency parameters are presented for the four cases with material constants for Mat.4. These differences are located between 3.0 and 5.5 percentages, and it is observed that there are no significant differences for different modes, laminations and boundary conditions.

Figure 3 presents also the lowest five frequency parameters for different sets of boundary conditions. The differences stay within the range of 3.0 and 5.5 percent, and no significant difference is observed, except for the FFFF case. As previously mentioned, the FFFF plate shows three rigid body motions and the vibration modes presented take very different shapes from the other sets (CFFF, CSFF, SSSS, CCCC).
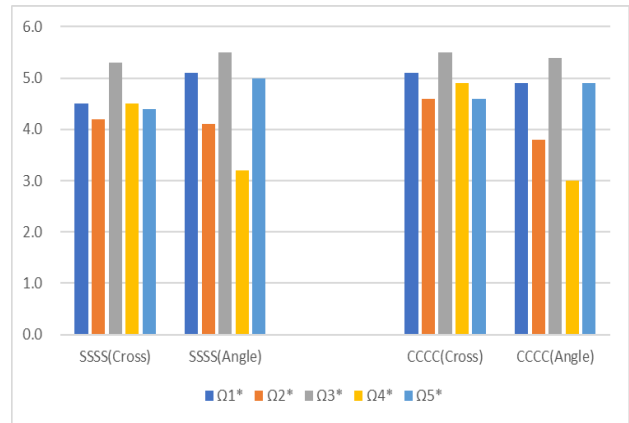

Figure 2. Comparison of percentage differences between cross-ply and angle-ply square plates with SSSS and CCCC boundary conditions (Mat.4)
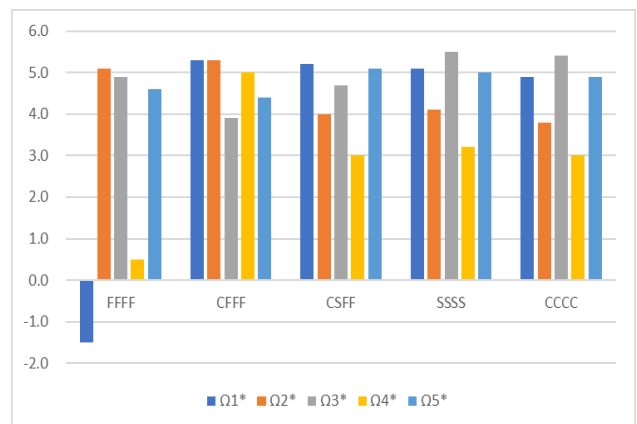

Figure 3. Comparison of percentage differences in angle-ply square plates among five sets of boundary conditions (Mat.4)
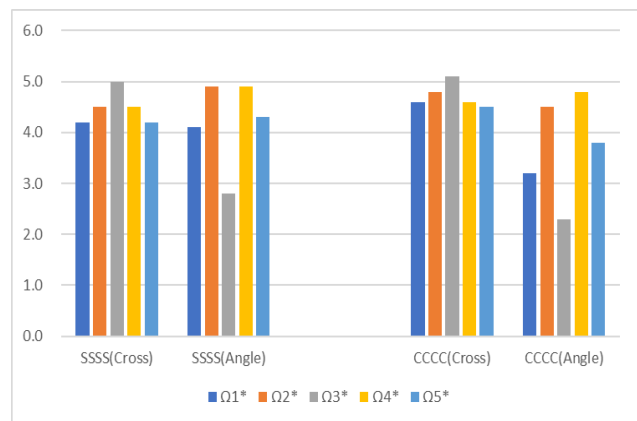

Figure 4. Comparison of percentage differences between cross-ply and angle-ply rectangular plates with SSSS and CCCC boundary conditions (Mat.4)(*a/b*=2)

Table 9. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer rectangular plate (a/b=2, SSSS)

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]$s				
Mat.1	95.97	159.0	296.2	350.8	383.9
(%)	−6.0	−6.6	−6.9	−5.9	−6.0
Mat.2	99.70	167.7	310.7	359.3	398.8
(%)	−2.3	−1.5	−2.4	−3.7	−2.3
Mat.3	97.96	163.5	306.0	357.9	391.8
(%)	−4.0	−3.9	−3.8	−4.1	−4.0
Mat.4	106.4	177.8	334.3	389.9	425.6
(%)	4.2	4.5	5.0	4.5	4.2
Mat.5	102.1	170.2	318.3	373.0	408.3
Angle-ply	$[(30/-30)_2]$s				
Mat.1	96.63	192.1	265.3	330.1	390.5
(%)	−5.7	−6.4	−4.6	−6.6	−5.8
Mat.2	99.32	197.7	273.8	340.9	401.8
(%)	−3.0	−3.7	−1.5	−3.5	−3.1
Mat.3	98.22	197.1	265.9	339.4	397.7
(%)	−4.1	−4.0	−4.3	−3.9	−4.1
Mat.4	106.7	215.3	285.6	370.7	432.4
(%)	4.1	4.9	2.8	4.9	4.3
Mat.5	102.4	205.2	277.9	353.3	414.6

Table 10. Frequency parameters Ω^* (Eq.(15)) of symmetric 8-layer rectangular plates (a/b=2, CCCC)

|	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
Cross-ply	$[(0/90)_2]$s				
Mat.1	209.9	279.4	427.7	546.8	586.2
(%)	−6.1	−6.5	−6.9	−6.0	−6.0
Mat.2	215.3	290.1	445.4	558.7	604.3
(%)	−3.6	−2.9	−3.0	−3.9	−3.2
Mat.3	214.4	287.0	441.4	558.0	598.8
(%)	−4.0	−3.9	−3.9	−4.1	−4.0
Mat.4	233.8	313.1	482.7	608.4	651.9
(%)	4.6	4.8	5.1	4.6	4.5
Mat.5	223.4	298.8	459.2	581.5	623.9
Angle-ply	$[(30/-30)_2]$s				
Mat.1	164.4	272.6	384.1	431.7	512.2
(%)	−4.9	−6.1	−4.2	−6.5	−5.4
Mat.2	169.9	281.8	397.2	447.2	528.3
(%)	−1.7	−3.0	−0.9	−3.1	−2.4
Mat.3	165.6	278.8	383.1	443.5	518.9
(%)	−4.3	−4.0	−4.4	−4.0	−4.2
Mat.4	178.4	303.4	409.9	483.8	561.7
(%)	3.2	4.5	2.3	4.8	3.8
Mat.5	172.9	290.4	400.8	461.7	541.4
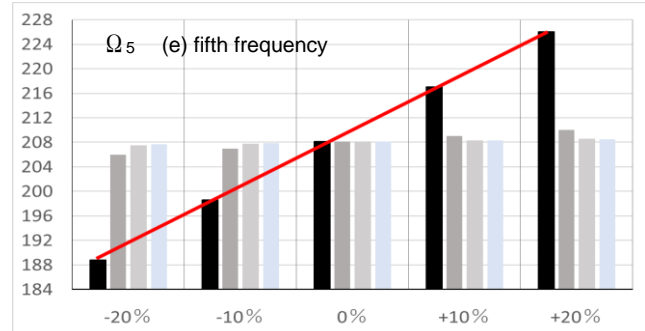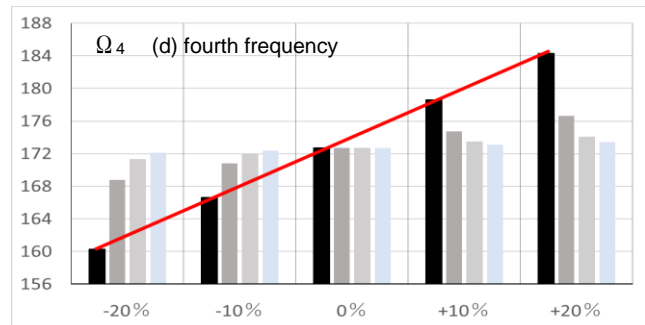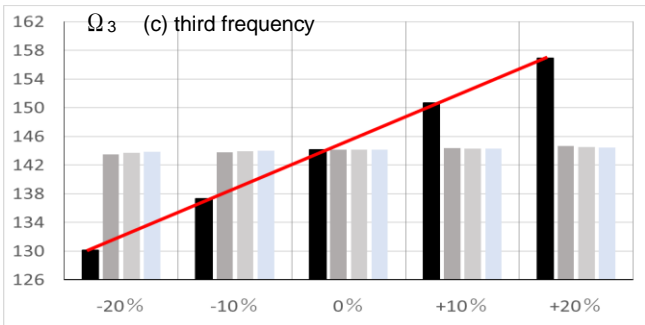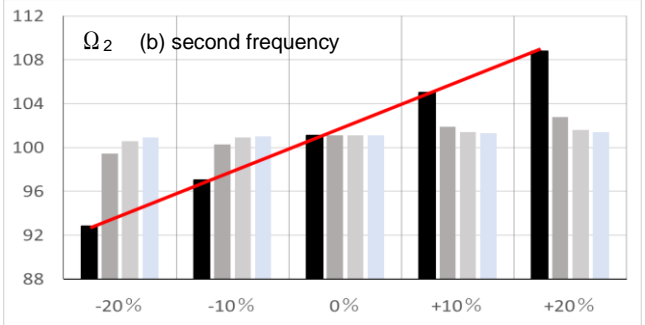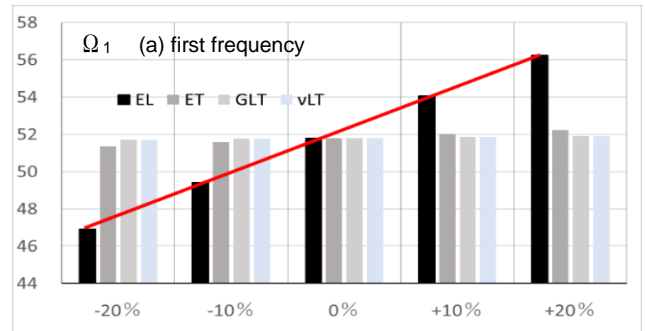
Figure 5. Variations of the lowest three frequency parameter with change (-20%～+20%) of the four elastic constants (E_L, E_T, G_{LT}, ν_{LT}) for square plate (SSSS) with $[(30/-30)_2]$s lamination

4.2. Effect of aspect ratio on frequency parameters

Numerical experiment is extended to include the frequency parameters of rectangular plate (a/b=2) to see the effect of aspect ratio. Table 9 present the lowest five frequency parameters of cross-ply and angle-ply rectangular plates with SSSS boundary conditions. The results in the same format is given in Table 10 for CCCC boundary condition. The data for Mat.4 is plotted in Fig.4 with the same format in Fig.2. When one compares results in Fig.4 with those in Fig.2, there is no difference globally and one can conclude that the differences do not exist for different aspect ratios.

4.3. Effect of each material constant on frequency parameters

From the results in Tables 4-8, it is obvious that Young's modulus E_L in the fiber direction is the most decisive controlling factor among the four constants to determine the frequency parameters. Therefore, numerical experiment is done in Fig.5 to calculate the lowest five frequencies $\Omega_1 \sim \Omega_5$ for the angle-ply square plate (SSSS) [(30/-30)$_2$]s by changing independently each of the four elastic constants in the range of 80 %～120 % from the reference value of Mat.5 (i.e., E_L=150GPa, E_T=10GPa, G_{LT}=5GPa, ν_{LT}=0.3). In the figure, the values with change of E_L are denoted by black solid columns and those with other three constants E_T, G_{LT}, ν_{LT} are by lighter grey columns. As clearly seen, frequency values increase significantly in linear fashion with change of E_L, while those values for change of the other three constants stay almost unchanged. Thus, the straight line can be used to approximate the frequency values, as indicated by red straight lines in the figure, for laminated plates with different CFRP materials under specified lamination condition of fixed aspect ratio and lamination scheme (in this case, cross-ply or angle-ply laminates).

A simple linear regression is applied by using

$$\Omega_i^* = C_1 \left(E_L / 10^9 \right) + C_0 \qquad (17)$$

where C_1 and C_0 indicate slope (sensitivity) and constant (intercept), respectively. Those constants are listed in Table 11 for two boundary conditions (SSSS, CCCC) and aspect ratios (a/b=1,2) of angle-ply plates [(30/-30)$_2$]s. By using these coefficients, approximate frequency parameters are evaluated by Eq. (17) for Mat.1-Mat.4 of square plates (SSSS, CCCC), and are compared to the values from Table 4 and 8. The values of difference in percentage are also listed in the table to assess accuracy of the formula. Except for three cases (Ω_2 and Ω_4 for Mat.3, and Ω_4 for Mat.4) of 20 frequencies presented, absolute values in discrepancies are less than one percent, and the accuracy of formula is demonstrated. Physically speaking, it is verified that Young's modulus E_L is dominant explanatory variable in the frequency evaluation of laminated plates.

Table 11. Coefficients of frequency formula in Eq. (17) for square and rectangular (a/b=2) plates [(30/-30)$_2$]s

		Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
SSSS	(a/b=1)					
C_1		0.155	0.266	0.446	0.401	0.616
C_0		28.38	61.06	76.86	112.31	115.27
CCCC	(a/b=1)					
C_1		0.257	0.363	0.641	0.506	0.797
C_0		49.80	88.94	112.50	149.50	153.68
SSSS	(a/b=2)					
C_1		0.272	0.603	0.595	1.040	1.122
C_0		61.53	114.45	188.35	196.46	245.70
CCCC	(a/b=2)					
C_1		0.393	0.803	0.782	1.330	1.351
C_0		113.70	169.50	283.10	261.25	337.98

Table 12. Comparison of frequency parameters between the formula estimation (17) and fully computed values for square plates [(30/-30)$_2$]s

	Ω_1^*	Ω_2^*	Ω_3^*	Ω_4^*	Ω_5^*
SSSS (a/b=1)					
Mat.1	48.37	95.45	133.8	164.4	194.4
Eq. (17)	48.22	95.11	133.9	163.6	194.1
(%)	−0.3	−0.4	0.1	−0.5	−0.2
Mat.2	49.89	98.11	139.2	168.8	200.8
Eq. (17)	49.77	97.77	138.4	167.6	200.3
(%)	−0.2	−0.3	−0.6	−0.7	−0.2
Mat.3	49.77	96.96	138.6	165.4	200.0
Eq. (17)	49.93	98.03	138.9	168.0	200.9
(%)	0.3	1.1	0.2	1.6	0.5
Mat.4	54.45	105.2	152.0	178.3	218.5
Eq. (17)	54.57	106.0	152.2	180.1	219.4
(%)	0.2	0.8	0.1	1.0	0.4
CCCC (a/b=1)					
Mat.1	82.66	135.8	194.2	214.9	255.8
Eq. (17)	82.70	135.40	194.5	214.3	255.7
(%)	0.0	−0.3	0.2	−0.3	0.0
Mat.2	85.84	140.2	202.4	221.4	265.2
Eq. (17)	85.27	139.0	201.0	219.3	263.7
(%)	−0.7	−0.9	−0.7	−0.9	−0.6
Mat.3	85.04	137.7	201.1	215.9	263.1
Eq. (17)	85.52	139.4	201.6	219.8	264.5
(%)	0.6	1.2	0.2	1.8	0.5
Mat.4	92.85	149.2	220.4	232.4	287.3
Eq. (17)	93.23	150.3	220.8	235.0	288.4
(%)	0.4	0.7	0.2	1.1	0.4

5. Conclusions

It has been demonstrated that, for symmetrically laminated CFRP rectangular plates having slightly different elastic constants, there is a underlying relation among calculated values of frequency parameters when the frequency parameters are properly defined. The Ritz method was used as an analytical tool in numerical experiments for solving free vibration of the plates. In numerical results, examples were given for cross-ply and angle-ply square plates. The frequency parameters for four different sets of materials were calculated and compared to those of new hypothetical material with the averaged and discretized constants. It turned out that, despite the difference of CFRP materials, the frequency parameters show unified behaviors with the change of Young's modulus in the fiber direction. It was demonstrated that it is feasible to derive approximate formulas to simultaneously predict the frequency parameters for laminated plates composed of different CFRP materials. The coefficients in the linear formula were tabulated and used to estimate the frequency parameters for limited cases, and it is hoped that this formula will be extended and used in many design situations.

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