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# Accurate Results for Free Vibration of Doubly Curved Shallow Shells of Rectangular Planform (Part. 2 Thickness effect)

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**Abstract**

This paper presents a follow-up study of a previous work that deals with the free vibration of moderately thin isotropic shallow shells under general edge conditions. The same semi-analytical method is used in this study for identical shape and degree of curvature in doubly curved geometry, and accurate natural frequencies are tabulated for a wide range of the shell edge conditions. Emphasis is made, however, to present the frequency parameters for the shallow shells with very thin thickness (representative length/shell thickness=100). In numerical experiments, convergence test is made against series terms in the case of very thin shallow shells. Twenty-one sets of frequency parameters are tabulated for three shell shapes (spherical, cylindrical and hyperbolic paraboloidal shells) and two curvature ratios. These two papers (Part. 1 and 2) will constitute the accurate standard in the area of shallow shell vibration of rectangular planform and serve for future comparison and practical design purpose.

*Keywords: Accuracy; free vibration; natural frequency; shallow shell; small thickness*

## 1. Introduction

There has been active and increasing usage of shallow shell structures in mechanical and structural engineering. A growth in the literature on shallow shell vibration has reflected this technical trend [1]. This author has, however, noticed in the published literature a significant lack of comprehensive sets of accurate natural frequencies to cover a wide range of shallow shell geometries and boundary conditions.

In a previous work [2], an attempt was made to present a semi-analytical method and to provide comprehensive lists of natural frequencies of open shallow shells of rectangular planform. As distinctive feature, not found in other types of closed shells, open shallow shell can take wide variations of geometric form, such as spherical shell, cylindrical shell and parabolic hyperbolic shell, each with different degree of curvature. In other words, there are a large number of combinations in the shape parameters.

For shallow shell vibration, relevant previous works are summarized in Ref. [2]. Practically the first landmark paper on this topic is one [3] published by Leissa and Kadi that formulates the exact solution of shallow shells of rectangular planform supported along four edges by shear diaphragm. This paper is followed by other works [4-7] in the 1980's, and by many works [8-18] up to the present. From the viewpoint of providing comprehensive lists of natural frequencies for general boundary conditions, it is noted that two papers [19, 20] present methods and numerical results under various combinations of in-plane and out-of-plane boundary conditions. In Ref. [19], Mochida and his co-workers use a superposition method and present natural frequencies of various shallow shells, but shallow shells with free edges are not included. Qatu and Asadi [20] present frequencies of the shells with twenty-one different sets of boundary conditions, but it seems to the present author that the numerical results are not well converged.

The objective of this work (Part. 2) is to present comprehensive lists of accurate frequency parameters of very thin shallow shells for twenty-one sets of the boundary conditions. With the two studies, reasonably sufficient free vibration information can be summarized to cover shallow shells with relatively thick case (representative edge length/shell thickness=20) in Part. 1 [2] and very thin case in Part. 2 (thickness ratio=100). The convergence of the solution and comparison with other methods are severely checked, and effect of thickness is discussed by comparing these sets of results.

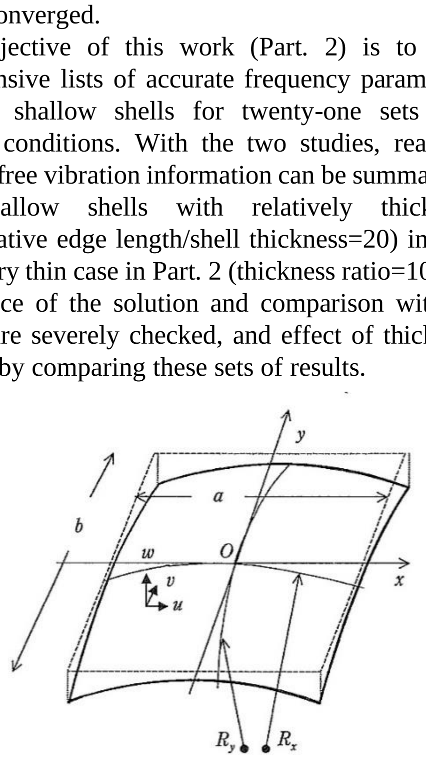


Figure 1. Shallow shell in the coordinate system

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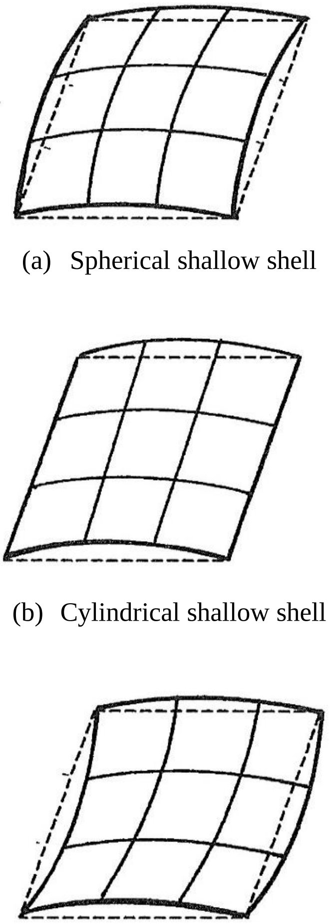

(a) Spherical shallow shell

(b) Cylindrical shallow shell

(c) Hyperbolic paraboloidal shallow shell

Figure 2. Shallow shells of rectangular planform

## 2. Outline of Analytical Method

The geometry of quadratic mid-surface can be expressed for a doubly curved shallow shell in a rectangular coordinate system (see Fig. 1) by

$$\phi(x,y) = -\frac{1}{2}\left(\frac{x^2}{R_x} + \frac{y^2}{R_y}\right) \quad (1)$$

where $R_x$ and $R_y$ are the radii of curvature in the $x$ and $y$ directions, respectively. The dimension of its planform is given by $a \times b$ and the thickness is $h$. The four sides are subjected to uniform in-plane (i.e., stretching) and out-of-plane (bending) boundary conditions.

This shell takes geometric form of a spherical shell for $1/R_x$=$1/R_y$=(finite) in Fig. 2(a), and takes form of a cylindrical shell for "$1/R_x$=(finite) and $1/R_y$=0 ($R_y$=∞)" or "$1/R_y$=(finite) and $1/R_x$=0 ($R_x$=∞)" in Fig. 2(b). When positive curvature exists in $x$ direction and negative curvature in $y$ direction, or vice versa, it takes form of a hyperbolic paraboloidal shell for $1/R_x$= -$1/R_y$=(finite) in Fig. 2(c).

In the previous study [2], details of extended Ritz method are presented based on Donnell-type shallow shell theory. The same method is used here. The stretching, stretching-bending coupling and bending stiffness matrices are given, respectively, by

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \quad (2a)$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \quad (2b)$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \quad (2c)$$

For isotropic material, they are simply reduced to

$$A_{ij} = hQ_{ij} \ , \ B_{ij} = 0 \ , \ D_{ij} = \left(\frac{h^3}{12}\right) Q_{ij} \quad (3)$$

($i,j$=1,2,6), where the coefficients are elastic constants

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, \ Q_{12} = \nu Q_{11}, \ Q_{66} = G \quad (4)$$

Here, $E$ is the modulus of elasticity, $G=E/2(1+\nu)$ is the shear modulus and ν is Poisson's ratio.

This semi-analytical method requires the evaluation of energy functional

$$L = T_{max} - \left(V_{s,max} + V_{bs,max} + V_{b,max}\right) \quad (5)$$

where $V_s$ , $V_{bs}$ and $V_b$ are the parts of the total strain energy due to stretching, bending- stretching coupling and bending, respectively, and $T$ is translational kinetic energy. The stationary value is determined in the functional by

$$\frac{\partial L}{\partial P_{ij}} = 0, \ \frac{\partial L}{\partial Q_{kl}} = 0, \ \frac{\partial L}{\partial R_{mn}} = 0 \quad (6)$$

$(i,k,m = 0,1,2,..,(M-1);\ j,l,n = 0,1,2,...,(N-1))$

where $P_{ij}$ , $Q_{kl}$ and $R_{mn}$ are unknown coefficients in the displacement functions. The displacement functions are formulated to satisfy at least the kinematical boundary conditions along the edges. Use of boundary index makes it possible to accommodate any combination of in-plane and out-of-plane boundary conditions [2].

After applying the process in Eq. (6), frequency equation is derived as

$$det\left([K] - \Omega^2 [M]\right) = 0 \quad (7)$$

where [$K$] and [$M$] are global stiffness and mass (inertia) matrices, respectively. The Ω is a frequency parameter

$$\Omega = \omega a^2 \sqrt{\frac{\rho h}{D}} \text{ (dimensionless frequency)} \quad (8)$$

$$D = Eh^3 / 12\left(1-\nu^2\right) \text{ (reference plate stiffness)} \quad (9)$$

The lowest six eigenvalues from Eq.(7) are frequency parameters to be listed in the following tables. It should be noted again that arbitrary sets of boundary conditions can be specified, and details are given in previous study [2]. For edge condition, "F" and ""C" indicate all four displacements are unconstrained and constrained, respectively, and "S" does simply support with in-plane displacement parallel to the edge is zero but one perpendicular to the edge is unconstrained [2].

## 3. Numerical Examples and Accuracy of Solution

### *3.1. Convergence and comparison of the solution*

In the present numerical examples, very thin thickness (*a/h*=100) is assumed to study the thickness effect by comparing the results with those in previous study [2] for relatively thick case (*a/h*=20). Square planform (*a/b*=1), except in Table.2, and Poisson's ratio =0.3 are used.

Table 1 presents convergence study of frequency parameters of spherical ($R_x/R_y$=1), cylindrical ($R_x/R_y$=0, i.e., $R_y$=∞) and hyperbolic paraboloidal ($R_x/R_y$= -1) shells of square planform. The shells (SSSS) in this table are supported by shear diaphragm along four edges, and the exact solution is available [3]. For each shell configuration, two degrees of curvature $a/R_x$=0.2 and 0.5 are used. The present results are calculated for the number of terms 8×8, 10×10 and 12×12 for each of *u*, *v* and *w*, and as in the relatively thick shell [2], the present parameters similarly converge well within five significant figures, and are in very good agreement with the exact values. A pair of half wave number of out-of-plane displacement *w* is given by (*m,n*) in the table.

Table 1 Convergence and comparison of frequency parameters Ω of simply supported shallow shells (SSSS), *a/b* =1, *a/h* =100, ν =0.3.

| | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| Spherical shell (*R*x/*R*y=1, *a/R*x=0.2) | | | | | | |
| 8×8 | 68.858 | 82.426 | 82.426 | 102.92 | 118.77 | 118.77 |
| 10×10 | 68.857 | 82.425 | 82.425 | 102.92 | 118.74 | 118.74 |
| 12×12 | 68.857 | 82.425 | 82.425 | 102.92 | 118.74 | 118.74 |
| Exact | 68.858 | 82.425 | 82.425 | 102.92 | 118.74 | 118.74 |
| (m,n) | (1,1) | (1,2) | (2,1) | (2,2) | (1,3) | (3,1) |
| Spherical shell (*R*x/*R*y=1, *a/R*x=0.5) | | | | | | |
| 8×8 | 164.63 | 171.70 | 171.70 | 182.64 | 192.09 | 192.09 |
| 10×10 | 164.63 | 171.70 | 171.70 | 182.63 | 192.05 | 192.05 |
| 12×12 | 164.63 | 171.70 | 171.70 | 182.63 | 192.05 | 192.05 |
| Exact | 164.63 | 171.70 | 171.70 | 182.63 | 192.05 | 192.05 |
| (m,n) | (1,1) | (1,2) | (2,1) | (2,2) | (1,3) | (3,1) |
| Cylindrical shell (*R*x/*R*y=0, *a/R*x=0.2) | | | | | | |
| 8×8 | 38.437 | 51.061 | 72.299 | 85.563 | 98.921 | 115.24 |
| 10×10 | 38.437 | 51.059 | 72.299 | 85.562 | 98.893 | 115.22 |
| 12×12 | 38.437 | 51.059 | 72.299 | 85.562 | 98.892 | 115.22 |
| Exact | 38.437 | 51.059 | 72.299 | 85.562 | 98.893 | 115.22 |
| (m,n) | (1,1) | (1,2) | (2,1) | (2,2) | (3,1) | (1,3) |
| Cylindrical shell (*R*x/*R*y=0, *a/R*x=0.5) | | | | | | |
| 8×8 | 59.191 | 84.179 | 99.987 | 114.02 | 137.86 | 140.79 |
| 10×10 | 59.184 | 84.179 | 99.915 | 114.02 | 137.81 | 140.79 |
| 12×12 | 59.184 | 84.179 | 99.914 | 114.02 | 137.81 | 140.79 |
| Exact | 59.184 | 84.179 | 99.914 | 114.02 | 137.81 | 140.79 |
| (m,n) | (2,1) | (1,1) | (3,1) | (2,2) | (3,2) | (1,2) |
| Hyperbolic paraboloidal shell (*R*x/*R*y=-1, *a/R*x=0.2) | | | | | | |
| 8×8 | 19.660 | 63.238 | 63.238 | 78.879 | 111.95 | 111.95 |
| 10×10 | 19.659 | 63.236 | 63.236 | 78.877 | 111.93 | 111.93 |
| 12×12 | 19.660 | 63.236 | 63.236 | 78.877 | 111.93 | 111.93 |
| Exact | 19.660 | 63.236 | 63.236 | 78.877 | 111.93 | 111.93 |
| (m,n) | (1,1) | (1,2) | (2,1) | (2,2) | (1,3) | (3,1) |
| Hyperbolic paraboloidal shell (*R*x/*R*y=-1, *a/R*x=0.5) | | | | | | |
| 8×8 | 19.257 | 78.472 | 109.98 | 109.98 | 142.75 | 142.75 |
| 10×10 | 19.257 | 78.461 | 109.98 | 109.98 | 142.70 | 142.70 |
| 12×12 | 19.258 | 78.461 | 109.98 | 109.98 | 142.70 | 142.70 |
| Exact | 19.257 | 78.461 | 109.98 | 109.98 | 142.70 | 142.70 |
| (m,n) | (1,1) | (2,2) | (1,2) | (2,1) | (1,3) | (3,1) |

Table 2 is a comparison study with values of Ref. [19] by Mochida and his co-workers. They use the method of superposition that is known as a method to provide numerical solutions in good accuracy. They provide frequency parameters only for combinations of two types of in-plane constraints, where displacement normal to the edge is zero and displacement parallel to the edge is zero, and two type of out-of-plane displacement, i.e., simple support and clamped edge. In their work, no results are presented for cases involving free edges.

In a previous study [2], comparison is made also with their values, but for avoiding duplication in this work, different sets of boundary conditions are used. Also result for CCCC is given for a rectangle (*a/b*=0.5). It is found in the table that the present values exactly agree with their values, when the present ones are rounded with four significant figures. Accuracy of the present solution is

Table 2 Comparison of frequency parameters Ω of shallow shells, *a/Rx* =0.5, *a/h* =100, ν =0.3.

| | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| SCSS (*a/b* =1) | | | | | | |
| Spherical shell (*R*x/*R*y=1) | | | | | | |
| Present | 168.71 | 173.78 | 181.43 | 188.92 | 193.37 | 204.54 |
| Ref.[19] | 168.7 | 173.8 | 181.4 | 188.9 | 193.4 | 204.6 |
| Cylindrical shell (*R*x/*R*y=0) | | | | | | |
| Present | 64.987 | 87.899 | 102.51 | 120.76 | 144.25 | 144.86 |
| Ref.[19] | 64.98 | 87.90 | 102.5 | 120.8 | 144.2 | 144.9 |
| Hyperbolic paraboloidal shell (*R*x/*R*y=-1) | | | | | | |
| Present | 82.623 | 90.895 | 130.08 | 131.01 | 150.55 | 169.19 |
| Ref.[19] | 82.62 | 90.90 | 130.1 | 131.0 | 150.5 | 169.2 |
| SCSC (*a/b* =1) | | | | | | |
| Spherical shell (*R*x/*R*y=1) | | | | | | |
| Present | 177.61 | 182.78 | 185.75 | 195.19 | 196.05 | 219.24 |
| Ref.[19] | 177.6 | 182.8 | 185.8 | 195.2 | 196.1 | 219.2 |
| Cylindrical shell (*R*x/*R*y=0) | | | | | | |
| Present | 71.325 | 92.637 | 105.67 | 127.94 | 149.84 | 151.31 |
| Ref.[19] | 71.32 | 92.63 | 105.7 | 127.9 | 149.8 | 151.3 |
| Hyperbolic paraboloidal shell (*R*x/*R*y=-1) | | | | | | |
| Present | 127.84 | 133.29 | 135.91 | 142.72 | 170.73 | 171.02 |
| Ref.[19] | 127.9 | 133.3 | 135.9 | 142.7 | 170.7 | 171.0 |
| CCCC (*a/b* =0.5) | | | | | | |
| Spherical shell (*R*x/*R*y=1) | | | | | | |
| Present | 179.41 | 179.90 | 180.47 | 187.88 | 189.62 | 192.52 |
| Ref.[19] | 179.4 | 179.9 | 180.5 | 187.9 | 189.6 | 192.5 |
| Cylindrical shell (*R*x/*R*y=0) | | | | | | |
| Present | 72.277 | 95.799 | 106.45 | 116.43 | 120.96 | 132.05 |
| Ref.[19] | 72.27 | 95.78 | 106.4 | 116.4 | 120.9 | 132.0 |
| Hyperbolic paraboloidal shell (*R*x/*R*y=-1) | | | | | | |
| Present | 131.04 | 131.20 | 139.19 | 139.90 | 154.35 | 155.52 |
| Ref.[19] | 131.0 | 131.2 | 139.2 | 139.9 | 154.4 | 155.5 |

thus established, and the following results are obtained in using the 12×12 solution that are presented in five significant figures.

*3.2. Comprehensive results of shallow shells*

Table 3(a) presents the lowest six frequency parameters Ω of shallow spherical shell ($R_x/R_y$=1) of square planform ($a/b$=1) with very small thickness ($a/h$=100) for twenty-one sets of boundary conditions. The degree of curvature is taken as $a/R_x$=$a/R_y$=0.2. Table 3(b) has the same format as Table 3(a), except that the curvature is larger in $a/R_x$/=$a/R_y$=0.5. Addition of curvature causes frequencies to be increased. In Table 3(a), the average increase from flat plates for the fundamental frequencies in $\Omega_1$ is 105.5 percent, including the highest increase 20 percent of CSSS shell. These increases are much larger than the case of relatively thin shell ($a/h$=20) in [2], and roughly speaking, are almost ten times larger. One should note, however, that the frequency parameter defined in Eqs. (8) and (9) include the thickness $h$ explicitly, and direct comparison between two sets of the parameters with different thickness ratios ($a/h$=20 and 100) may not be appropriate. This will be discussed later in this paper.

In Table 3(b), the deeper curvature $a/R$=0.5 causes the average increase of 252 percent in $\Omega_1$ with the maximum 734 percent of SSSS shell, when they are compared to the flat plate frequencies.

Table 4(a) and (b) tabulate the lowest six frequency parameters of shallow cylindrical shells with $R_x$=(infinity) and $a/R_y$=0.2 and $a/R_y$=0.5, respectively. This shell takes straight edges of the shell along the $x$ axis, and curvature is given only along $y$ direction. When these tables are compared with those of spherical shell in Tab. 3(a) and (b), the effect of unidirectional curvature is a half of the spherical shells, and generally the effect of curvature increase in one direction is a half of curvatures in two directions of spherical shells.

Table 5(a) and (b) also tabulate the lowest six frequency parameters of shallow cylindrical shells, but with $R_y$=(infinity) and $a/R_x$=0.2 and $a/R_x$=0.5, respectively. Straight edges of the shell exist along the $y$ axis, and curvature exists only along $x$ direction. Likewise in results in [2], For cylindrical shells with FFFF, SSSS and CCCC, the frequency values are the identical as in Table 4(a) and (b) due to uniform boundary condition along four edges. Similarly, cylindrical shells with SSFF, CCFF and CCSS give the identical results as in Table 4(a) and (b) since the 90 degree rotation of the shell gives essentially the same boundary conditions. The same results in six cases are underlined.

Table 6(a) and (b) list up the lowest six parameters of shallow hyperbolic paraboloidal shells with $a/R_y$=0.2 and $a/R_y$=0.5, respectively. Just like in [2], the negative curvature ratio ($R_x/R_y$= -1) gives rise unusual response in frequency. Namely, for shell of hyperbolic paraboloidal shell, negative curvature causes decrease of frequencies, when shell has free edges. Four cases among twenty-one, the shell gives lower frequencies than frequencies of flat plate given in Table 7. As the constrained is increased along the edges, this tendency disappears.

Table 3(a) Frequency parameters Ω of shallow spherical shells, $R_x/R_y$=1, $a/R_x$=$a/R_y$=0.2, $a/b$=1, $a/h$=100, $\nu$=0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.521 | 19.753 | 35.878 | 35.878 | 42.331 | 69.570 |
| SFFF | 6.6712 | 17.335 | 26.490 | 35.783 | 61.072 | 61.267 |
| CFFF | 6.5822 | 8.8500 | 24.876 | 32.141 | 38.908 | 68.677 |
| SSFF | 3.4094 | 18.027 | 28.038 | 57.296 | 66.252 | 74.143 |
| CSFF | 8.4529 | 21.169 | 33.288 | 60.520 | 74.132 | 78.481 |
| CCFF | 16.271 | 26.213 | 43.803 | 68.672 | 79.180 | 83.149 |
| SFSF | 12.424 | 16.938 | 47.450 | 49.994 | 75.167 | 93.448 |
| CFSF | 21.671 | 37.610 | 58.211 | 64.713 | 77.247 | 100.30 |
| SSSF | 14.395 | 48.653 | 71.242 | 86.457 | 90.390 | 98.601 |
| CSSF | 31.481 | 61.465 | 74.028 | 92.204 | 93.843 | 113.78 |
| CCSF | 31.773 | 61.515 | 79.182 | 97.294 | 101.32 | 113.82 |
| CFCF | 61.320 | 61.437 | 69.751 | 74.382 | 82.536 | 103.95 |
| SCSF | 15.689 | 48.666 | 76.431 | 90.101 | 98.635 | 99.237 |
| CSCF | 61.391 | 71.924 | 80.512 | 94.731 | 102.02 | 127.78 |
| CCCF | 61.426 | 71.952 | 87.036 | 104.06 | 105.42 | 132.40 |
| SSSS | 68.857 | 82.425 | 82.425 | 102.92 | 118.74 | 118.74 |
| CSSS | 72.522 | 84.443 | 90.352 | 109.41 | 120.20 | 131.99 |
| CCSS | 76.255 | 91.412 | 93.796 | 115.75 | 133.02 | 133.89 |
| CSCS | 80.177 | 87.332 | 98.107 | 117.07 | 122.04 | 147.14 |
| CCCS | 84.183 | 96.662 | 100.01 | 123.14 | 135.28 | 148.86 |
| CCCC | 96.717 | 102.74 | 102.74 | 130.31 | 148.49 | 154.07 |

Table3(b) Frequency parameters Ω of shallow spherical shells, $R_x/R_y$=1, $a/R_x$=$a/R_y$=0.5, $a/b$=1, $a/h$=100, $\nu$=0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.574 | 19.983 | 36.858 | 36.858 | 49.526 | 70.625 |
| SFFF | 6.6235 | 17.914 | 27.470 | 38.759 | 64.412 | 71.594 |
| CFFF | 9.0041 | 9.7546 | 30.402 | 33.937 | 49.019 | 71.839 |
| SSFF | 3.3656 | 18.488 | 30.212 | 61.617 | 78.160 | 123.22 |
| CSFF | 12.169 | 23.168 | 42.208 | 67.551 | 88.251 | 128.57 |
| CCFF | 21.593 | 32.629 | 68.321 | 78.368 | 110.04 | 140.52 |
| SFSF | 13.116 | 17.328 | 53.474 | 54.929 | 110.64 | 110.95 |
| CFSF | 31.378 | 53.208 | 73.001 | 98.427 | 127.89 | 138.31 |
| SSSF | 14.987 | 54.190 | 110.80 | 166.05 | 173.40 | 175.80 |
| CSSF | 48.905 | 87.488 | 132.54 | 169.13 | 177.43 | 180.71 |
| CCSF | 49.647 | 87.592 | 132.58 | 174.10 | 182.71 | 188.86 |
| CFCF | 89.330 | 98.185 | 114.66 | 114.86 | 172.38 | 173.84 |
| SCSF | 20.617 | 54.193 | 110.80 | 172.06 | 176.98 | 182.58 |
| CSCF | 93.828 | 114.76 | 173.01 | 180.06 | 184.06 | 186.22 |
| CCCF | 93.835 | 114.77 | 173.26 | 184.38 | 189.33 | 195.28 |
| SSSS | 164.63 | 171.70 | 171.70 | 182.63 | 192.05 | 192.05 |
| CSSS | 168.71 | 173.78 | 181.43 | 188.92 | 193.37 | 204.54 |
| CCSS | 171.60 | 180.27 | 186.99 | 195.32 | 204.04 | 210.09 |
| CSCS | 177.61 | 182.78 | 185.75 | 195.19 | 196.05 | 219.24 |
| CCCS | 179.90 | 187.88 | 192.50 | 202.47 | 208.02 | 228.73 |
| CCCC | 191.99 | 191.99 | 196.93 | 209.96 | 216.19 | 242.22 |

Table 4(a) Frequency parameters Ω of shallow cylindrical shells, $R_x$ =(infinity), $a/R_y$ =0.2, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.483 | 21.903 | 34.850 | 37.642 | 38.468 | 61.117 |
| SFFF | 6.6792 | 25.137 | 27.266 | 29.361 | 53.625 | 62.120 |
| CFFF | 8.3592 | 8.9008 | 26.822 | 33.231 | 35.102 | 58.709 |
| SSFF | 3.3876 | 18.187 | 28.597 | 49.805 | 52.146 | 63.917 |
| CSFF | 9.5159 | 21.294 | 34.312 | 52.845 | 54.308 | 73.867 |
| CCFF | 11.007 | 29.916 | 35.356 | 62.297 | 65.099 | 74.651 |
| SFSF | 17.370 | 19.560 | 39.632 | 50.600 | 52.218 | 75.590 |
| CFSF | 22.359 | 25.244 | 43.750 | 60.101 | 61.232 | 77.857 |
| SSSF | 19.168 | 36.550 | 51.489 | 62.571 | 73.691 | 97.950 |
| CSSF | 24.151 | 40.977 | 60.740 | 65.130 | 80.745 | 104.66 |
| CCSF | 24.529 | 50.278 | 60.808 | 77.810 | 85.915 | 113.99 |
| CFCF | 28.583 | 31.660 | 48.648 | 70.829 | 71.612 | 80.604 |
| SCSF | 19.697 | 46.614 | 51.545 | 75.749 | 79.201 | 99.759 |
| CSCF | 30.225 | 45.975 | 68.284 | 71.291 | 89.195 | 112.37 |
| CCCF | 30.533 | 54.804 | 71.342 | 80.478 | 94.018 | 121.44 |
| SSSS | 38.437 | 51.059 | 72.299 | 85.562 | 98.892 | 115.22 |
| CSSS | 41.773 | 54.061 | 79.051 | 92.603 | 100.63 | 127.85 |
| CCSS | 48.570 | 67.079 | 81.803 | 100.93 | 116.03 | 129.48 |
| CSCS | 45.944 | 57.772 | 87.426 | 100.72 | 102.78 | 142.05 |
| CCCS | 51.987 | 69.788 | 90.013 | 108.44 | 117.90 | 143.61 |
| CCCC | 67.681 | 78.294 | 94.610 | 116.46 | 135.00 | 145.76 |

Table 4(b) Frequency parameters Ω of shallow cylindrical shells, $R_x$ =(infinity), $a/R_y$ =0.5, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.507 | 22.073 | 34.868 | 48.702 | 54.308 | 61.193 |
| SFFF | 6.7626 | 25.729 | 34.960 | 41.383 | 64.227 | 79.883 |
| CFFF | 10.588 | 16.980 | 30.638 | 42.203 | 47.659 | 65.439 |
| SSFF | 3.3702 | 18.226 | 37.799 | 52.273 | 79.338 | 83.783 |
| CSFF | 14.358 | 32.691 | 44.477 | 54.022 | 86.787 | 91.932 |
| CCFF | 14.983 | 42.872 | 49.155 | 73.302 | 89.760 | 96.679 |
| SFSF | 22.529 | 29.269 | 61.011 | 64.354 | 65.594 | 77.232 |
| CFSF | 29.383 | 34.058 | 71.553 | 73.837 | 74.882 | 80.774 |
| SSSF | 25.569 | 65.016 | 66.151 | 71.716 | 116.30 | 116.48 |
| CSSF | 31.551 | 70.922 | 74.274 | 79.607 | 118.44 | 123.64 |
| CCSF | 31.933 | 74.264 | 76.427 | 108.31 | 131.26 | 132.00 |
| CFCF | 36.942 | 39.735 | 82.233 | 84.268 | 84.971 | 85.003 |
| SCSF | 26.099 | 65.100 | 70.344 | 104.38 | 118.41 | 124.46 |
| CSCF | 38.287 | 76.311 | 84.646 | 87.429 | 121.03 | 131.22 |
| CCCF | 38.528 | 82.521 | 84.699 | 113.45 | 138.63 | 143.98 |
| SSSS | 59.184 | 84.179 | 99.914 | 114.02 | 137.81 | 140.79 |
| CSSS | 64.986 | 87.899 | 102.51 | 120.76 | 144.25 | 144.86 |
| CCSS | 72.375 | 105.48 | 127.24 | 133.02 | 147.49 | 168.62 |
| CSCS | 71.325 | 92.637 | 105.67 | 127.94 | 149.84 | 151.31 |
| CCCS | 78.068 | 108.33 | 134.01 | 134.72 | 153.32 | 173.98 |
| CCCC | 99.263 | 119.00 | 151.13 | 156.35 | 172.52 | 192.43 |

Table 5(a) Frequency parameters Ω of shallow cylindrical shells, $R_y$=(infinity), $a/R_x$=0.2, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.483 | 21.903 | 34.850 | 37.642 | 38.468 | 61.117 |
| SFFF | 6.6303 | 15.153 | 25.367 | 38.286 | 49.549 | 59.200 |
| CFFF | 3.4740 | 8.4730 | 21.673 | 30.718 | 38.552 | 61.123 |
| SSFF | 3.3876 | 18.187 | 28.597 | 49.805 | 52.146 | 63.917 |
| CSFF | 5.3711 | 23.986 | 29.189 | 58.193 | 62.305 | 65.517 |
| CCFF | 11.007 | 29.916 | 35.356 | 62.297 | 65.099 | 74.651 |
| SFSF | 9.7012 | 16.074 | 38.965 | 46.722 | 62.804 | 75.604 |
| CFSF | 15.183 | 36.307 | 49.399 | 61.353 | 66.142 | 86.169 |
| SSSF | 11.672 | 41.185 | 51.040 | 62.204 | 83.750 | 90.277 |
| CSSF | 26.655 | 53.502 | 56.075 | 74.830 | 86.146 | 105.96 |
| CCSF | 30.034 | 54.215 | 59.721 | 78.205 | 94.146 | 106.29 |
| CFCF | 58.315 | 60.258 | 61.144 | 70.589 | 73.150 | 96.003 |
| SCSF | 19.980 | 42.029 | 54.733 | 66.491 | 90.640 | 91.840 |
| CSCF | 58.926 | 63.964 | 67.510 | 84.225 | 89.974 | 118.23 |
| CCCF | 59.122 | 64.590 | 70.454 | 87.422 | 97.705 | 125.27 |
| SSSS | 38.437 | 51.059 | 72.299 | 85.562 | 98.892 | 115.22 |
| CSSS | 45.859 | 64.956 | 75.151 | 94.571 | 114.52 | 116.94 |
| CCSS | 48.570 | 67.079 | 81.803 | 100.93 | 116.03 | 129.48 |
| CSCS | 63.541 | 73.706 | 80.634 | 103.59 | 119.33 | 131.90 |
| CCCS | 65.171 | 75.764 | 86.896 | 109.46 | 131.64 | 133.36 |
| CCCC | 67.681 | 78.294 | 94.610 | 116.46 | 135.00 | 145.76 |

Table 5(b) Frequency parameters Ω of shallow cylindrical shells, $R_y$=(infinity), $a/R_x$=0.5, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.507 | 22.073 | 34.868 | 48.702 | 54.308 | 61.193 |
| SFFF | 6.5659 | 15.172 | 25.255 | 48.942 | 51.902 | 59.303 |
| CFFF | 3.4483 | 8.2887 | 21.400 | 29.527 | 51.692 | 60.834 |
| SSFF | 3.3702 | 18.226 | 37.799 | 52.273 | 79.338 | 83.783 |
| CSFF | 5.3196 | 23.755 | 38.350 | 62.825 | 80.388 | 89.447 |
| CCFF | 14.983 | 42.872 | 49.155 | 73.302 | 89.760 | 96.679 |
| SFSF | 9.6518 | 15.726 | 38.968 | 46.600 | 87.930 | 95.924 |
| CFSF | 14.858 | 46.682 | 49.066 | 90.447 | 102.91 | 106.34 |
| SSSF | 11.528 | 41.099 | 76.463 | 90.188 | 109.96 | 111.74 |
| CSSF | 39.866 | 69.777 | 85.768 | 109.29 | 117.32 | 140.36 |
| CCSF | 46.746 | 74.034 | 90.546 | 109.96 | 120.78 | 147.64 |
| CFCF | 60.841 | 85.618 | 101.77 | 106.69 | 129.75 | 137.60 |
| SCSF | 39.743 | 43.622 | 82.174 | 90.776 | 112.50 | 115.78 |
| CSCF | 69.222 | 103.23 | 112.13 | 124.49 | 156.98 | 158.29 |
| CCCF | 71.051 | 104.02 | 115.45 | 127.91 | 163.30 | 163.49 |
| SSSS | 59.184 | 84.179 | 99.914 | 114.02 | 137.81 | 140.79 |
| CSSS | 67.289 | 103.11 | 120.99 | 131.59 | 142.40 | 164.20 |
| CCSS | 72.375 | 105.48 | 127.24 | 133.02 | 147.49 | 168.62 |
| CSCS | 92.641 | 114.28 | 140.12 | 145.75 | 168.92 | 182.49 |
| CCCS | 95.798 | 116.43 | 145.24 | 150.67 | 170.30 | 187.31 |
| CCCC | 99.263 | 119.00 | 151.13 | 156.35 | 172.52 | 192.43 |

Table 6(a) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, $R_x/R_y$= -1, $a/R_y$=0.2, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.462 | 24.738 | 36.955 | 36.955 | 52.563 | 63.898 |
| SFFF | 6.6380 | 21.641 | 26.605 | 43.730 | 55.348 | 63.658 |
| CFFF | 6.4996 | 8.7953 | 29.914 | 32.664 | 46.110 | 64.991 |
| SSFF | 3.3656 | 20.411 | 38.508 | 39.592 | 63.965 | 66.883 |
| CSFF | 7.6241 | 28.830 | 42.288 | 52.961 | 65.428 | 75.539 |
| CCFF | 8.6992 | 33.048 | 50.320 | 63.530 | 74.322 | 78.713 |
| SFSF | 16.528 | 16.753 | 50.102 | 54.907 | 56.478 | 70.884 |
| CFSF | 26.302 | 37.366 | 59.804 | 64.066 | 64.665 | 84.047 |
| SSSF | 17.199 | 38.042 | 54.112 | 63.519 | 77.934 | 95.426 |
| CSSF | 33.152 | 48.479 | 64.771 | 76.666 | 81.357 | 105.33 |
| CCSF | 41.686 | 56.645 | 65.357 | 83.766 | 92.208 | 115.14 |
| CFCF | 61.217 | 63.091 | 70.497 | 73.361 | 74.448 | 94.428 |
| SCSF | 34.797 | 46.701 | 54.358 | 72.630 | 89.198 | 100.62 |
| CSCF | 62.678 | 66.374 | 74.210 | 86.152 | 87.457 | 114.99 |
| CCCF | 63.201 | 71.908 | 74.659 | 92.908 | 97.131 | 124.50 |
| SSSS | 19.660 | 63.236 | 63.236 | 78.877 | 111.93 | 111.93 |
| CSSS | 41.441 | 68.114 | 74.443 | 91.002 | 114.11 | 125.71 |
| CCSS | 52.386 | 77.989 | 79.478 | 100.61 | 127.67 | 127.81 |
| CSCS | 65.750 | 77.903 | 83.122 | 100.84 | 117.48 | 141.59 |
| CCCS | 69.936 | 86.409 | 87.335 | 109.50 | 130.71 | 143.33 |
| CCCC | 79.599 | 94.110 | 94.110 | 117.77 | 145.72 | 145.92 |

Table 6(b) Frequency parameters Ω of shallow hyperbolic paraboloidal shells, $R_x/R_y$= -1, $a/R_y$=0.5, $a/b$ =1, $a/h$ =100, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.424 | 25.664 | 38.909 | 38.909 | 64.227 | 79.301 |
| SFFF | 6.5797 | 23.152 | 28.121 | 48.293 | 72.323 | 88.044 |
| CFFF | 8.2199 | 9.4072 | 36.242 | 36.427 | 72.068 | 81.721 |
| SSFF | 3.3264 | 21.403 | 38.639 | 72.319 | 82.019 | 94.838 |
| CSFF | 8.9011 | 34.299 | 75.355 | 80.171 | 88.219 | 112.27 |
| CCFF | 9.8110 | 38.144 | 85.107 | 95.412 | 110.78 | 127.38 |
| SFSF | 17.348 | 18.533 | 60.794 | 71.367 | 107.19 | 111.31 |
| CFSF | 43.390 | 58.519 | 89.161 | 102.31 | 113.00 | 123.96 |
| SSSF | 18.753 | 64.362 | 73.281 | 96.120 | 111.74 | 122.48 |
| CSSF | 66.052 | 71.660 | 107.99 | 112.92 | 125.04 | 141.99 |
| CCSF | 77.735 | 94.160 | 109.74 | 126.07 | 145.85 | 151.05 |
| CFCF | 98.601 | 116.10 | 121.74 | 127.34 | 128.31 | 140.44 |
| SCSF | 73.294 | 74.852 | 88.379 | 98.575 | 123.21 | 146.89 |
| CSCF | 105.30 | 124.20 | 130.30 | 140.39 | 141.58 | 146.16 |
| CCCF | 106.46 | 126.11 | 137.44 | 143.08 | 165.67 | 166.80 |
| SSSS | 19.258 | 78.461 | 109.98 | 109.98 | 142.70 | 142.70 |
| CSSS | 82.623 | 90.895 | 130.08 | 131.01 | 150.55 | 169.19 |
| CCSS | 94.181 | 122.38 | 136.98 | 149.24 | 173.01 | 173.62 |
| CSCS | 127.84 | 133.29 | 135.91 | 142.72 | 170.73 | 171.02 |
| CCCS | 131.20 | 139.90 | 154.35 | 158.42 | 186.02 | 189.83 |
| CCCC | 157.35 | 157.35 | 157.41 | 166.52 | 204.03 | 208.69 |

Table 7 Frequency parameters Ω of flat square plates, $a/b$ =1, $\nu$ =0.3.

| B.C. | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ | $\Omega_6$ |
|---|---|---|---|---|---|---|
| FFFF | 13.468 | 19.596 | 24.270 | 34.801 | 34.801 | 61.093 |
| SFFF | 6.6433 | 14.902 | 25.376 | 26.001 | 48.449 | 50.579 |
| CFFF | 3.4711 | 8.5065 | 21.286 | 27.199 | 30.958 | 54.189 |
| SSFF | 3.3674 | 17.316 | 19.293 | 38.211 | 51.035 | 53.487 |
| CSFF | 5.3512 | 19.076 | 24.671 | 43.089 | 52.707 | 63.762 |
| CCFF | 6.9200 | 23.907 | 26.586 | 47.655 | 62.709 | 65.537 |
| SFSF | 9.6313 | 16.135 | 36.726 | 38.945 | 46.738 | 70.740 |
| CFSF | 15.192 | 20.584 | 39.736 | 49.449 | 56.280 | 77.324 |
| SSSF | 11.685 | 27.756 | 41.197 | 59.065 | 61.861 | 90.294 |
| CSSF | 16.792 | 31.114 | 51.397 | 64.021 | 67.540 | 101.12 |
| CCSF | 17.537 | 36.023 | 51.812 | 71.077 | 74.326 | 105.79 |
| CFCF | 22.168 | 26.407 | 43.597 | 61.176 | 67.176 | 79.817 |
| SCSF | 12.687 | 33.065 | 41.702 | 63.015 | 72.398 | 90.611 |
| CSCF | 23.371 | 35.571 | 62.875 | 66.762 | 77.374 | 108.87 |
| CCCF | 23.921 | 39.998 | 63.221 | 76.710 | 80.572 | 116.66 |
| SSSS | 19.739 | 49.348 | 49.348 | 78.957 | 98.696 | 98.696 |
| CSSS | 23.646 | 51.674 | 58.646 | 86.134 | 100.27 | 113.23 |
| CCSS | 27.054 | 60.538 | 60.786 | 92.836 | 114.56 | 114.70 |
| CSCS | 28.951 | 54.743 | 69.327 | 94.585 | 102.22 | 129.10 |
| CCCS | 31.826 | 63.331 | 71.076 | 100.79 | 116.36 | 130.35 |
| CCCC | 35.985 | 73.394 | 73.394 | 108.22 | 131.58 | 132.20 |

### *3.3. Discussion on thickness effect*

Although the frequency parameter Ω defined in Eqs. (8) and (9) has been used for shallow shell vibration in the past literature, it turns out in this study that direct comparison is not appropriate because thickness $h$ is included in the parameter Ω. A new frequency parameter is therefore proposed here as

$$\Omega^* = \Omega\left(\frac{h}{a}\right) = \omega a\sqrt{\frac{12\rho\left(1-\nu^2\right)}{E}} \tag{10}$$

that is still nondimensional and proportional to ω, but excludes thickness ratio ($a/h$) in the parameter. In other words, comparison of $\Omega^*$ is more reasonable between two cases with different thickness.

Figure 3 illustrates variations of new frequency parameter Ω* for three lowest modes of spherical shell versus four different thickness ratios of $a/h$=10 (thick shell), 20, 100 and 1000 (extremely thin shell). Theoretically, $a/h$=10 might be almost limit of applicable range of the thin shell theory. Figure 3(a) represents small curvature of $a/R_x=a/R_y$=0.2, and lower figure (b) does large curvature $a/R_x=a/R_y$=0.5. Values of $\Omega^*$ are inserted for $a/h$=10 and 20 in each figure, but omitted for $a/h$=100 and 1000 due to lack of space. It is clearly seen in both figures that all frequency parameters Ω* monotonically decrease with decreasing bending stiffness, but interestingly the difference due to curvature increase from $a/R_x=a/R_y$=0.2 to $a/R_x=a/R_y$=0.5 is not significant.

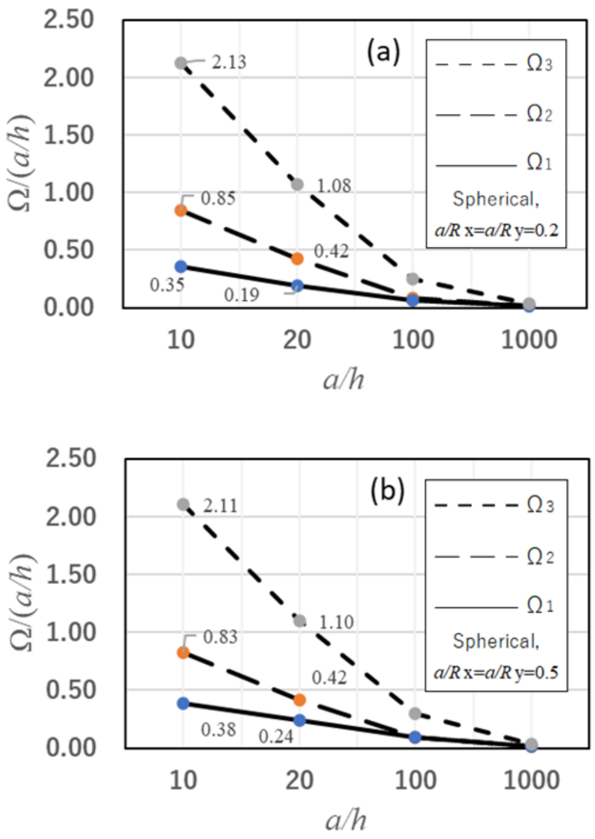


Figure 3. Thickness effect (Spherical shell, CFFF)

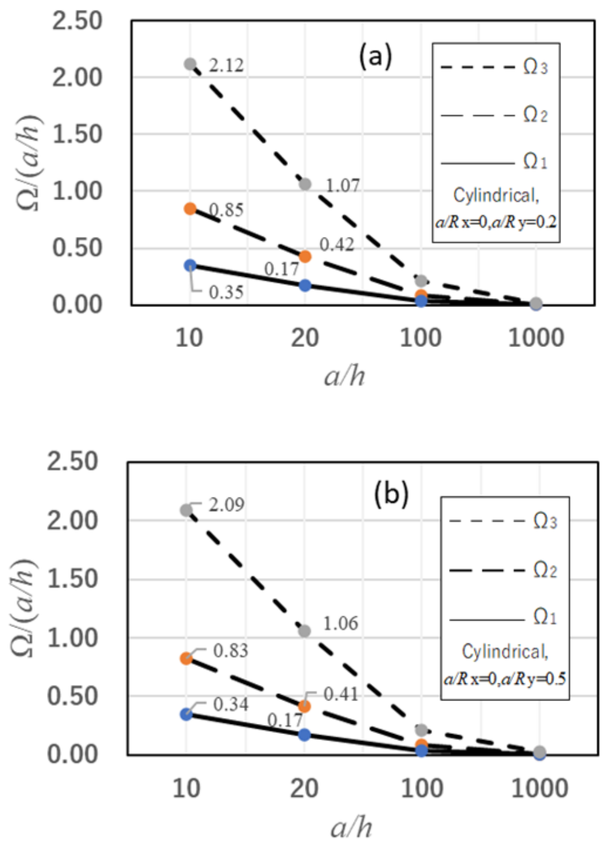


Figure 4. Thickness effect (Cylindrical shell, CFFF)

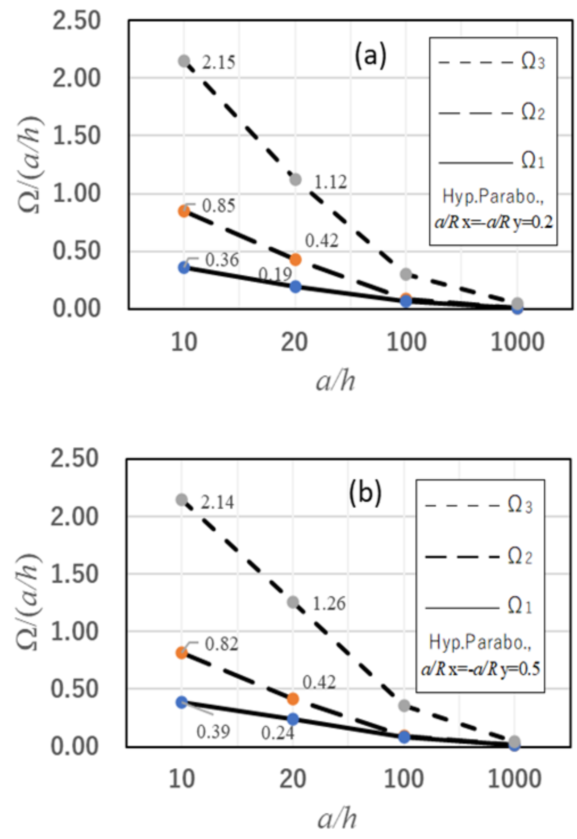


Figure 5. Thickness effect (Hyp.Para. shell, CFFF)

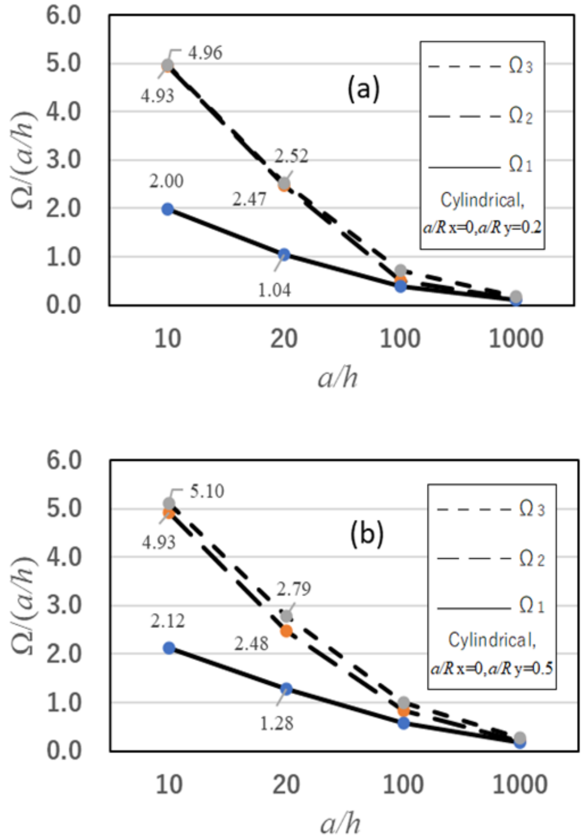


Figure 6. Thickness effect (Cylindrical shell, SSSS)

Figures 4 and 5 present in the same format the results of frequency $\Omega^*=\Omega(h/a)$ for cylindrical shell and hyperbolic paraboloidal shell, respectively. When these six sets of variations in $\Omega^*$ in Fig.3-5 are compared, the difference in frequencies stays within a small range of parameter, for example, the first frequency parameter changes between $\Omega_1^*=0.34$ and 0.39 for $a/h=10$, and does between $\Omega_1^*=0.17$ and 0.24 for $a/h=20$. Generally speaking, these six figures present quite similar forms of variation.

Figure 6 illustrates the variation of $\Omega_1^*$ for cylindrical shell with totally simply supported edges (SSSS). Figures 6(a) and (b) show that frequency behavior and values of the parameters differ from shell with cantilever type boundary condition (CFFF) in in Fig.3-5, but the frequency decrease takes similar tendency.

## 4. Conclusions

As Part. 2 with a previous study (Part. 1) [2], this paper tabulated accurate natural frequencies for free vibration of doubly curved, isotropic shallow shells of rectangular (square) planform, when the shell thickness is small (representative length/shell thickness=100). Twenty-one different sets of boundary conditions are included. The same mathematical procedure was used and was briefly outlined.

In the process of this study, it was found that the traditional representation in frequency parameter may not be appropriate to evaluate the effect of changing shell thickness on free vibration of the shells. Based on this finding, a new frequency parameter was proposed by excluding thickness in the parameter. With this new parameter, rather unified behavior was identified, and more effective use of this parameter will be studied.

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